THE COLONEL BLOTTO GAME WITH APPLICATIONS TO THE ECONOMIC, MILITARY AND POLITICAL SCIENCES

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To my wife Kasie, always and forever.

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TABLE OF CONTENTS

Page

ABSTRACT vi

1 The Colonel Blotto Game 1

1.1 The Model 2

1.2 Optimal Univariate Marginal Distributions 4

1.3 Existence of Sufficient n-copulas 9

1.4 Conclusion 21

1.5 Bibliography 23

1.6 Appendix 24

2 Fiscal Federalism and the Incentives of Redistributive Politics 27

2.1 The Model 31

2.1.1 Centralized Political Competition 32

2.1.2 Decentralized Political Competition 33

2.2 Results 35

2.2.1 Optimal Univariate Marginal Distributions 35

2.2.2 Existence of Sufficient n-copulas 42

2.2.3 Centralization vs. Decentralization 45

2.3 Conclusion 48

2.4 Bibliography 50

2.5 Appendix 51

3 Electoral Poaching and Party Identification
Joint Work with Dan Kovenock 55

3.1 The Model 60

3.2 Transformations of the Electorate 67

3.3 Conclusion 78

Page

3.4 Bibliography 80

3.5 Appendix 81

VITA 86

ABSTRACT

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In the Colonel Blotto game, two players simultaneously distribute forces across n battlefields. Within each battlefield, the player that allocates more force wins. The payoff of the game is the proportion of the wins on the individual battlefields. An equilibrium of the Colonel Blotto game consists of a pair of n-variate distributions. Chapter 1 demonstrates how to separate the players' best response correspondences into a set of univariate marginal distributions and a mapping of this set into an n-variate distribution; fully characterizes the equilibrium univariate marginal distributions for this class of games; and constructs corresponding equilibrium n-variate distributions.

Chapter 2 compares centralized to decentralized electoral competition in a model of redistributive politics with local public goods. In this setting, the level of inequality arising from each party's equilibrium redistribution schedule is higher in a centralized system. In addition, if the utilities provided by the local public goods are above a minimal threshold, then centralization is also found to create greater inefficiencies in the provision of the local public goods. However, the inefficiency of centralization is due to the targetability of local public goods and the ability to share resources across jurisdictions not to interjurisdictional externalities or heterogeneities in the production of or preferences for local public goods.

Chapter 3 examines electoral competition in a model of redistributive politics with heterogeneous voter loyalties to political parties. We construct a natural measure of "party strength" based on the sizes and intensities of a party's loyal voter segments and demonstrate how party behavior varies with the two parties' strengths.

In equilibrium, parties target or "poach" a strict subset of the opposition party's loyal voters: offering those voters a high expected transfer, while "freezing out" the remainder with a zero transfer. The size of the subset of opposition voters frozen out and, consequently, the level of inequality in a party's equilibrium redistribution schedule is increasing in the opposition party's strength. We also construct a measure of "political polarization" that is increasing in the sum and symmetry of the parties' strengths, and find that the inequality of the implemented policy is increasing in political polarization.

1. The Colonel Blotto Game

The Colonel Blotto game, which originates with Borel (1921), is a constant-sum game involving two players, A and B, and n independent battlefields. A has X_A units of force to distribute among the battlefields, and B has X_B units. Each player must distribute their forces without knowing the opponent's distribution. If A sends x_A^k units and B sends x_B^k units to the kth battlefield, the player who provides the higher level of force wins battlefield k. The payoff for the whole game is the proportion of the wins on the individual battlefields. The first solution of this game appears in Borel and Ville (1938) who solve the problem for the case of $n = 3$ and $X_A = X_B$. Gross and Wagner (1950) extend this solution to allow for any finite $n \geq 3$, but still require that $X_A = X_B$.

This paper extends the literature on the Colonel Blotto game by completely characterizing the equilibrium univariate marginal distributions. Since the appearance of the solution to the symmetric case, it has been an open question whether uniform univariate marginal distributions are a necessary condition for equilibrium.[1] We show that the answer to this question is yes. To characterize the equilibrium univariate marginal distributions, we utilize n-copulas, the functions that map univariate marginal distributions into joint distributions, to separate the players' best response correspondences into a set of univariate marginal distributions and a mapping of this set into an n-variate distribution.[2] The characterization of the equilibrium univariate marginal distributions presented here also allows us to extend the Colonel Blotto game by allowing the players to have asymmetric forces.

The Colonel Blotto game is a fundamental model of strategic resource allocation in multiple dimensions. Strategic resource allocation in a single dimension, such

[1]See for example Gross and Wagner (1950) and Laslier and Picard (2002) who discuss this issue.
[2]See Nelsen (1999) for an introduction to copulas.

as the all-pay auction, has been widely used in economics to model contests such as political campaigns, political lobbying, research and development races, litigation and a number of other applications. Most if not all of these applications have multiple dimension analogs. In addition, the Colonel Blotto game has recently been used to analyze electoral competition over redistribution of a fixed budget (Laslier and Picard (2002), Laslier (2002)). In the model of redistributive politics candidates simultaneously announce how they will allocate a budget, if elected, by making binding promises to each voter. Each voter votes for the candidate offering the higher level of utility, and each candidate's payoff is the vote share that they receive. The Colonel Blotto game with asymmetric forces, examined in this paper, corresponds directly to a model of redistributive politics in which one candidate has a valence advantage.

Section 2 presents the model. Section 3 completely characterizes the equilibrium univariate marginal distributions of the Colonel Blotto game. Section 4 demonstrates the existence of n-copulas with the necessary properties. Section 5 concludes.

1.1 The Model

Players

Two players, A and B, simultaneously allocate their forces, X_A and X_B respectively, across a finite number, $n \geq 3$, of homogeneous battlefields.[3] Each battlefield j has a payoff of $\frac{1}{n}$. Each player's payoff is the sum of the payoffs across all of the battlefields or, equivalently, the proportion of the battlefields to which the player sends a higher level of force. Let $X_A \leq X_B$. In the case that the players allocate the same level of

[3]The case of $n = 2$, with symmetric and asymmetric forces, is discussed by Gross and Wagner (1950). Moving from $n = 2$ to $n \geq 3$ greatly enlarges the space of feasible n-variate distribution functions, and the equilibrium strategies examined in this paper differ dramatically from the case of $n = 2$.

force to a battlefield, player B wins that battlefield.[4] The force allocated to each battlefield must be nonnegative. For player i, the set of feasible allocations of force across the n battlefields is denoted by

$$\mathfrak{B}_i = \left\{ \mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^{n} x^j \leq X_i \right\}.$$

Strategies

It is well known that for $\frac{1}{n}X_B \leq X_A \leq X_B$ there is no pure strategy equilibrium for this class of games.[5] A mixed strategy, which we term a *distribution of force*, for player i is an n-variate distribution function $P_i : \mathbb{R}^n_+ \rightarrow [0, 1]$ with support contained in the set of player i's feasible allocations of force, $\mathfrak{B}_i$, and with one-dimensional marginal distribution functions $\left\{F_i^j\right\}_{j \in \{1,\ldots,n\}}$, one univariate marginal distribution function for each battlefield j. The n-tuple of player i's allocation of force across the n battlefields is a random n-tuple drawn from the n-variate distribution function P_i with the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^{n}$.

The Colonel Blotto game

The Colonel Blotto game, which we label

$$CB\left\{X_A, X_B, n\right\},$$

is the one-shot game in which players compete by simultaneously announcing distributions of force, subject to their budget constraints, each battlefield is won by the player that provides the higher allocation of force on that battlefield, where player B wins the battlefield in the case that both players allocate the same level of force to that battlefield, and players' payoffs equal the proportion of battles won.

[4] The specification of the tie-breaking rule does not affect the results as long as $\frac{2}{n}X_B \leq X_A$. In the case that $\frac{2}{n}X_B > X_A$, this specification of the tie-breaking rule ensures weak lower semicontinuity of the players' best response correspondences and hence an equilibrium (see Dasgupta and Maskin (1986)).

[5] In the case that $\frac{1}{n}X_B > X_A$, there, trivially, exists a pure strategy equilibrium, and player B wins all of the battlefields.

1.2 Optimal Univariate Marginal Distributions

To completely characterize the equilibrium univariate marginal distribution functions, we utilize n-copulas, the functions that map univariate marginal distribution functions into joint distribution functions.

Definition: Let I denote the unit interval $[0,1]$. An *n-copula* is a function C from I^n to I such that

1. For all $\mathbf{x} \in I^n$, $C(\mathbf{x}) = 0$ if at least one coordinate of $\mathbf{x}$ is 0, and if all coordinates of $\mathbf{x}$ are 1 except x_k, then $C(\mathbf{x}) = x_k$.
2. For every $\mathbf{x}$, $\mathbf{y} \in I^n$ such that $x_k \leq y_k$ for all $k \in \{1, \ldots, n\}$, the C-volume of the n-box $[x_1, y_1] \times \ldots \times [x_n, y_n]$,

$$V_C([\mathbf{x}, \mathbf{y}]) = \Delta_{x_n}^{y_n} \Delta_{x_{n-1}}^{y_{n-1}} \ldots \Delta_{x_2}^{y_2} \Delta_{x_1}^{y_1} C(\mathbf{t})$$

where

$$\begin{aligned} \Delta_{x_k}^{y_k} C(\mathbf{t}) &= C(t_1, \ldots, t_{k-1}, y_k, t_{k+1}, \ldots, t_n) \\ &\quad - C(t_1, \ldots, t_{k-1}, x_k, t_{k+1}, \ldots, t_n) \end{aligned}$$

is greater than or equal to 0.

Given the definition of an n-copula, we can state the crucial property of n-copulas that we will use.

Theorem 1 [Sklar's Theorem in n-dimensions]: Let H be an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$. Then there exists an n-copula C such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \tag{1.1}$$

Conversely, if C is an n-copula and $F_1, F_2, \ldots, F_n$ are univariate distribution functions, then the function H defined by equation 1 is an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$.

The proof of the two-dimensional version of Sklar's theorem is due to Sklar (1959). For a proof of the n-dimensional version see Schweizer and Sklar (1983).

One additional definition that will be used throughout the paper is the support of an n-variate distribution.

Definition: The *support of an n-variate distribution function*, H, is the complement of the union of all open sets of $\mathbb{R}^n$ with H-volume zero.

We now show that the univariate marginal distribution functions and the n-copula are separate components of the players' best response correspondences.

Proposition 1: Let $\frac{1}{n}X_B \leq X_A \leq X_B$. In $CB\{X_A, X_B, n\}$ each player's best response correspondence can be separated into the univariate marginal distribution functions and n-copula components. In particular, for a given P_{-i} the Lagrangian of each player's optimization problem[6] can be written as

$$\max_{\{F_i^j\}_{j=1}^n} \sum_{j=1}^{n} \left[\lambda_i \int_0^\infty \left[\frac{1}{n\lambda_i} F_{-i}^j(x) - x \right] dF_i^j \right] + \lambda_i X_i \tag{1.2}$$

where the set of univariate marginal distribution functions $\{F_i^j\}_{j=1}^n$ satisfy the constraint that there exists an n-copula, C, such that the support of the n-variate distribution $C(F_i^1(x^1), \ldots, F_i^n(x^n))$ is contained in $\left\{\mathbf{x} \in \mathbb{R}_+^n \mid \sum_{j=1}^n x^j = \sum_{j=1}^n E_{F_i^j}(x) = X_i\right\}$.

Proof: In the game $CB\{X_A, X_B, n\}$, for a given P_{-i} each player maximizes the sum of the expected payoffs across the individual battlefields

$$\max_{P_i} \sum_{j=1}^{n} \int_0^\infty \frac{1}{n} F_{-i}^j(x)\, dF_i^j$$

[6]This Lagrangian is for the case that for all battlefields both players' univariate marginal distributions do not place an atom on the same value. Clearly, in any optimal strategy this holds. However, it is straightforward to incorporate the tie-breaking rule into the Lagrangian of each player's optimization problem.

subject to the constraint that the support of the distribution of force P_i is contained in $\mathfrak{B}_i$ or equivalently the P_i-volume over the region $\left\{\mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^n x^j > X_i\right\}$ is 0.

Since $\frac{1}{n}X_B \leq X_A \leq X_B$, in any optimal strategy each player will allocate all of their forces with probability 1,

$$Pr_{P_i}\left[\sum_{j=1}^n x_i^j = X_i\right] = 1.$$

Let G_i denote the distribution function of $\sum_{j=1}^n x_i^j$ and note that $G_i(z)$ is the P_i-volume over the region $\left\{\mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^n x^j \leq z\right\}$. Since each player allocates all of their forces with probability 1, it follows that

$$G_i(z) = \begin{cases} 0 & if\ z < X_i \\ 1 & if\ z \geq X_i \end{cases}$$

Equivalently, the P_i-volume over the region $\left\{\mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^n x^j < X_i\right\}$ is 0. Since the P_i-volume over

$$\left\{\mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^n x^j < X_i\right\} \bigcup \left\{\mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^n x^j > X_i\right\}$$

is 0, the support of P_i is contained in $\left\{\mathbf{x} \in \mathbb{R}^n_+ | \sum_{j=1}^n x^j = X_i\right\}$.

In addition since each player allocates all of their forces with certainty, it follows directly that all of their forces are allocated in expectation, i.e. $E_{P_i}\left(\sum_{j=1}^n x_i^j\right) = X_i$. Let $\left\{F_i^j\right\}_{j=1}^n$ be the set of univariate marginal distribution functions of P_i. Finally, noting that

$$E_{P_i}\left(\sum_{j=1}^n x_i^j\right) = \sum_{j=1}^n E_{F_i^j}(x),$$

and that from Theorem 1 the n-variate distribution function P_i is equivalent to the set of univariate marginal distribution functions $\left\{F_i^j\right\}_{j=1}^n$ combined with an appropriate n-copula, C, the result follows directly. Q.E.D.

Note that from Theorem 1 an n-variate distribution function is equivalent to a set of univariate marginal distribution functions, $\{F_i^j\}_{j=1}^n$, and an n-copula, C. This in combination with the payoff function of this class of games allows us to separate the players' best response correspondences into the set of univariate marginal distribution functions and n-copula components. Moreover, contrary to the concerns stated by Gross and Wagner (1950), the existence of equilibrium n-variate distribution functions without a connected support is not problematic. Connectedness of the support is a property that arises from the n-copula. Proposition 1 makes no requirement on the connectedness of the resulting n-variate distribution function.[7] In particular, the only requirement on the set of feasible n-copulas is that given a set of equilibrium univariate marginal distribution functions, $\{F_i^j\}_{j=1}^n$, the combination of the n-copula and the set of equilibrium univariate marginal distribution functions must allocate all of the player's forces with probability 1.

We begin by completely characterizing the set of equilibrium univariate marginal distribution functions for $\frac{1}{n-1} \leq \frac{X_A}{X_B} \leq 1$ and then move on to constructing sufficient n-copulas. Theorem 2 examines the case of $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$ and Theorem 3 examines the case of $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$.[8] For the case that $\frac{1}{n} \leq \frac{X_A}{X_B} < \frac{1}{n-1}$, it is conjectured that the characterization of the equilibrium univariate marginal distributions given in Theorem 3 applies. The crucial issue for this parameter range is the existence of sufficient n-copulas, which is yet to be established.[9]

Theorem 2: Let X_A, X_B, and $n \geq 3$ satisfy $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$. The unique Nash equilibrium univariate marginal distribution functions of the game

[7]In fact, for $\frac{4}{n} \leq \frac{X_A}{X_B} \leq 1$ the n-variate distributions that are examined in Section 4 have disconnected supports. However, there are also sufficient n-variate distributions that have connected supports.

[8]The case that $\frac{1}{n}X_B > X_A$ is trivial.

[9]For $\frac{1}{n} \leq \frac{X_A}{X_B} < \frac{1}{n-1}$ and given the univariate marginal distributions in Theorem 3, the construction of sufficient n-copulas that is given in Section 4 provides a sufficient n-copula for player A but not for player B.

$CB\{X_A, X_B, n\}$ are for each player to allocate its forces according to the following univariate distribution functions. For player A

$$\forall\, j \in \{1,\ldots,n\} \quad F_A^j(x) = \left(1 - \tfrac{X_A}{X_B}\right) + \tfrac{x}{\frac{2}{n}X_B}\left(\tfrac{X_A}{X_B}\right) \quad x \in \left[0, \tfrac{2}{n}X_B\right]$$

Similarly for player B

$$\forall\, j \in \{1,\ldots,n\} \quad F_B^j(x) = \tfrac{x}{\frac{2}{n}X_B} \quad x \in \left[0, \tfrac{2}{n}X_B\right]$$

The expected payoff for player A is $\frac{X_A}{2X_B}$ and the expected payoff for player B is $1 - \frac{X_A}{2X_B}$.

The formal proof of Theorem 2 is given in the appendix. However, it is useful to provide some intuition for the uniqueness of the univariate marginal distribution functions.

Beginning with the characterization of n independent and identical simultaneous two-bidder all-pay auctions with complete information, let F_i^j represent bidder i's distribution of bids for auction j, and v_i^j represent the value of auction j for bidder i. Each bidder i's problem is

$$\max_{\{F_i^j\}_{j=1}^n} \sum_{j=1}^{n} \int_0^\infty \left[v_i^j F_{-i}^j(x) - x\right] dF_i^j.$$

Since each auction is independent, we can apply the equilibrium characterization of the single all-pay auction with complete information (see Hillman and Riley (1989) and Baye, Kovenock, and de Vries (1996)). Thus, there exists a unique equilibrium distribution function F_i^j for each auction j. For each auction j and bidder i we have the following three cases

$$\begin{array}{lll} If\ v_i^j > v_{-i}^j & F_i^j(x) = \frac{x}{v_{-i}^j} & x \in \left[0, v_{-i}^j\right] \\ If\ v_i^j = v_{-i}^j & F_i^j(x) = \frac{x}{v_i^j} & x \in \left[0, v_i^j\right] \\ If\ v_i^j < v_{-i}^j & F_i^j(x) = \left(\frac{v_{-i}^j - v_i^j}{v_{-i}^j}\right) + \frac{x}{v_{-i}^j} & x \in \left[0, v_i^j\right]. \end{array}$$

Now consider a Colonel Blotto game $CB\{X_A, X_B, n\}$. From Equation 2 in Proposition 1, each player's Lagrangian can be written as

$$\max_{\{F_i^j\}_{j=1}^n} \sum_{j=1}^{n} \left[\lambda_i \int_0^\infty \left[\frac{1}{n\lambda_i} F_{-i}^j(x) - x\right] dF_i^j\right] + \lambda_i X_i$$

subject to the constraint that there exists an n-copula, C, that results in an n-variate distribution $C\left(F_i^1\left(x^1\right),\ldots,F_i^n\left(x^n\right)\right)$ with support contained in $\left\{\mathbf{x}\in\mathbb{R}_+^n \middle| \sum_{j=1}^n x^j = X_i\right\}$. Assuming that a sufficient n-copula exists, the appendix establishes a one-to-one correspondence between the set of equilibrium univariate marginal distribution functions and the equilibrium distribution functions of bids from a unique set of n independent and identical simultaneous two-bidder all-pay auctions. Section 4, then, establishes the existence of sufficient n-copulas.

The following Theorem addresses the remaining case of $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$.

Theorem 3: Let X_A, X_B, and $n \geq 3$ satisfy $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$. The unique Nash equilibrium univariate marginal distribution functions of the game $CB\left\{X_A, X_B, n\right\}$ are for each player to allocate its forces according to the following univariate distribution functions. For player A

$$\forall\, j \in \{1,\ldots,n\} \quad F_A^j(x) = \left(1-\tfrac{2}{n}\right) + \tfrac{x}{X_A}\left(\tfrac{2}{n}\right) \quad x \in [0, X_A]$$

Similarly for player B

$$\forall\, j \in \{1,\ldots,n\} \quad F_B^j(x) = \begin{cases} \dfrac{2x\left(X_A-\frac{X_B}{n}\right)}{\left(X_A\right)^2} & x \in [0, X_A] \\ 1 & x \geq X_A \end{cases}$$

The expected payoff for player A is $\frac{2}{n} - \frac{2X_B}{n^2 X_A}$ and the expected payoff for player B is $1 - \frac{2}{n} + \frac{2X_B}{n^2 X_A}$.

The formal proof of Theorem 3 is similar to the proof contained in the appendix for Theorem 2, and is thus omitted.

1.3 Existence of Sufficient n-copulas

Subject to the constraint that there exist sufficient n-copulas, Theorems 2 and 3 characterize the unique sets of equilibrium univariate marginal distribution functions for $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$ and $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$, respectively. There is no known existence result for an n-copula, C, with the necessary property that, given a set of univariate marginal distribution functions $\{F_i^j\}_{j=1}^n$, the support of the n-variate distribution $C(F_i^1(x^1), \ldots, F_i^n(x^n))$ is contained in $\left\{\mathbf{x} \in \mathbb{R}_+^n | \sum_{j=1}^n x^j = \sum_{j=1}^n E_{F_i^j}(x) (= X_i)\right\}$. However from Theorem 1, it is sufficient to show that for each player there exists an n-dimensional distribution function that allocates all of that player's forces with probability 1 and that provides the unique sets of equilibrium univariate marginal distribution functions characterized in Theorems 2 and 3.

> **Theorem 4:** For each unique set of equilibrium univariate marginal distribution functions, $\{F_i^j\}_{j=1}^n$, characterized in Theorems 2 and 3, there exists an n-copula, C, such that the support of the n-variate distribution function $C(F_i^1(x^1), \ldots, F_i^n(x^n))$ is contained in $\left\{\mathbf{x} \in \mathbb{R}_+^n | \sum_{j=1}^n x^j = \sum_{j=1}^n E_{F_i^j}(x) (= X_i)\right\}$.

The rest of this section is devoted to a proof of this theorem. There are multiple n-variate distribution functions (and thus multiple n-copulas) that satisfy the necessary conditions. In the discussion that follows, we will focus on a new and novel way to construct sufficient n-variate distribution functions for this class of games. Recall that the ceiling function $\lceil x \rceil$ gives the smallest integer greater than or equal to x, and that the floor function $\lfloor x \rfloor$ gives the largest integer less than or equal to x. We begin with the case that $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$ as in Theorem 2. This proof is for player A. The proof for player B follows directly as the special case of player A where $\frac{X_A}{X_B} = 1$. The construction of the n-variate distribution function is outlined as follows.

1. Player A randomly selects $n - \left\lceil \frac{nX_A}{X_B} \right\rceil$ of the battlefields and provides zero forces to those battlefields.
2. If $\left\lceil \frac{nX_A}{X_B} \right\rceil - \left\lfloor \frac{nX_A}{X_B} \right\rfloor = 1$, then:
    (a) Player A randomly selects $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$ of the remaining $\left\lceil \frac{nX_A}{X_B} \right\rceil$ battlefields.
    (b) On the randomly selected $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$ battlefields, player A randomizes continuously on $\left[0, \frac{2}{n} X_B\right]$ on each of these battlefields such that, letting z be

the sum of player A's allocations of force on these $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$ battlefields and $G(z)$ be the distribution of z,

$$G(z) = \begin{cases} \frac{\left(z-\left(X_A - \frac{2}{n}X_B\right)\right)\left(1-\left\lceil \frac{nX_A}{X_B}\right\rceil + \frac{nX_A}{X_B}\right)}{\frac{2}{n}X_B} & z \in \left[X_A - \frac{2}{n}X_B, X_A\right) \\ 1 & z \geq X_A \end{cases}$$

(c) Defining the allocation of force on the remaining battlefield as $X_A - z$, it follows directly that the univariate distribution of force on the remaining battlefield places mass $\left\lceil \frac{nX_A}{X_B} \right\rceil - \frac{nX_A}{X_B}$ at 0 and randomizes continuously on $\left[0, \frac{2}{n}X_B\right]$ with the remaining mass. In addition for all realizations, $\mathbf{x} \in \mathbb{R}^n_+$, of this strategy $\sum_{j=1}^{n} x^j = X_A$ with probability 1.

(d) There are $\binom{n}{\left\lceil \frac{nX_A}{X_B} \right\rceil}\binom{\left\lceil \frac{nX_A}{X_B} \right\rceil}{1}$ ways of dividing n battlefields into disjoint $\left(n - \left\lceil \frac{nX_A}{X_B} \right\rceil\right)$-subsets, $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$-subsets, and 1-subsets. Constructing an n-variate distribution as described above on each of the $\binom{n}{\left\lceil \frac{nX_A}{X_B} \right\rceil}\binom{\left\lceil \frac{nX_A}{X_B} \right\rceil}{1}$ possible divisions of the n battlefields into disjoint $\left(n - \left\lceil \frac{nX_A}{X_B} \right\rceil\right)$-subsets, $\left\lfloor \frac{nX_A}{X_B} \right\rfloor$-subsets, and 1-subsets and weighting each of these n-variate distribution functions by $\left[\binom{n}{\left\lceil \frac{nX_A}{X_B} \right\rceil}\binom{\left\lceil \frac{nX_A}{X_B} \right\rceil}{1}\right]^{-1}$, the n-variate distribution function formed by taking the sum of these weighted n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\left(1 - \frac{X_A}{X_B}\right)$ at 0 and randomize continuously on $\left[0, \frac{2}{n}X_B\right]$.

3. If $\left\lceil \frac{nX_A}{X_B} \right\rceil - \left\lfloor \frac{nX_A}{X_B} \right\rfloor = 0$, then:

(a) On the remaining $\frac{nX_A}{X_B}$ battlefields, player A randomizes continuously on $\left[0, \frac{2}{n}X_B\right]$ on each of these battlefields such that, letting z be the sum of player A's allocations of force on these battlefields and $G(z)$ be the distribution of z,

$$G(z) = \begin{cases} 0 & z < X_A \\ 1 & z \geq X_A \end{cases}$$

Thus, for all realizations, $\mathbf{x} \in \mathbb{R}^n_+$, of this strategy $\sum_{j=1}^{n} x^j = X_A$ with probability 1.

(b) There are $\binom{n}{\frac{nX_A}{X_B}}$ ways of dividing the n battlefields into disjoint $\left(n - \frac{nX_A}{X_B}\right)$-subsets and $\frac{nX_A}{X_B}$-subsets. Constructing an n-variate distribution as described above on each of the $\binom{n}{\frac{nX_A}{X_B}}$ possible divisions of the n battlefields into disjoint $\left(n - \frac{nX_A}{X_B}\right)$-subsets and $\frac{nX_A}{X_B}$-subsets and weighting each of these n-variate distribution functions by $\left[\binom{n}{\frac{nX_A}{X_B}}\right]^{-1}$, the n-variate distribution function formed by taking the sum of these weighted n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\left(1 - \frac{X_A}{X_B}\right)$ at 0 and randomize continuously on $\left[0, \frac{2}{n}X_B\right]$.

The pivotal steps in this construction are points 2 (b) and 3 (a), and we will now show that there exist such multivariate distribution functions. Beginning with the case that $\frac{2}{n} \leq \frac{X_A}{X_B} \leq \frac{3}{n}$, from points 2 and 3 player A allocates force to at least two and not more than three battlefields, which we label battlefields 1, 2, and 3. Let x_i denote the allocation of force to battlefield $i \in \{1, 2, 3\}$, $z = x_2 + x_3$, and $x_1 = X_A - z$. Consider the support of a bivariate distribution, F, for x_2, x_3 which uniformly places mass $\frac{X_A}{\frac{2}{n}X_B} - 1$ on each of the two following line segments

1. $\left(\frac{2}{n}X_B, X_A - \frac{2}{n}X_B\right)$ to $\left(X_A - \frac{2}{n}X_B, 0\right)$
2. $\left(X_A - \frac{2}{n}X_B, \frac{2}{n}X_B\right)$ to $\left(0, X_A - \frac{2}{n}X_B\right)$

and uniformly places the remaining mass, $3 - \frac{nX_A}{X_B}$, on the line segment $\left(\frac{2}{n}X_B, X_A - \frac{2}{n}X_B\right)$ to $\left(X_A - \frac{2}{n}X_B, \frac{2}{n}X_B\right)$. This support is shown in Figure 1.

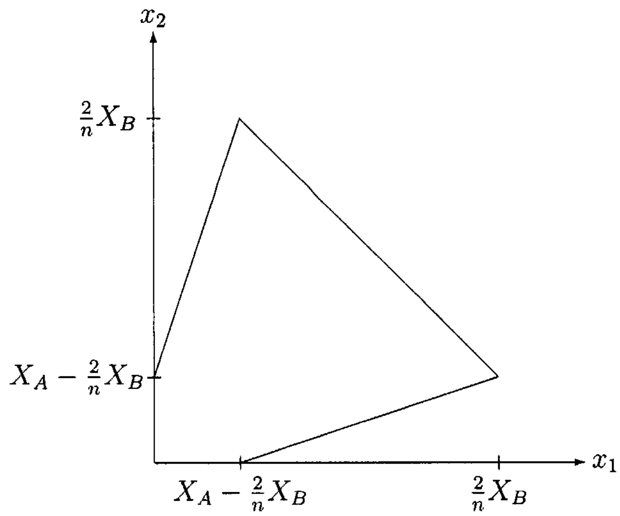


Figure 1: Support of the Bivariate Distribution F

In the expression for this bivariate distribution function we will use the following notation.

- R1: $\left\{(x_2, x_3) \in \left[0, \frac{2}{n}X_B\right]^2 | x_2 + x_3 > X_A\right\}$
- R2: $\left\{(x_2, x_3) \in \left[0, \frac{2}{n}X_B\right]^2 | x_2 > \frac{\frac{4}{n}X_B - X_A}{X_A - \frac{2}{n}X_B}x_1 + X_A - \frac{2}{n}X_B\right\}$
- R3: $\left\{(x_2, x_3) \in \left[0, \frac{2}{n}X_B\right]^2 | x_3 > \frac{\frac{4}{n}X_B - X_A}{X_A - \frac{2}{n}X_B}x_2 + X_A - \frac{2}{n}X_B\right\}$
- R4: $\left\{(x_2, x_3) \in \left[0, \frac{2}{n}X_B\right]^2 | x_2, x_3 < X_A - \frac{2}{n}X_B\right\}$
- R5: $\left\{(x_2, x_3) \in \left[0, \frac{2}{n}X_B\right]^2 | (x_2, x_3) \notin \text{R1} \cup \text{R2} \cup \text{R3} \cup \text{R4}\right\}$

The bivariate distribution function for x_2, x_3 is given by

$$F(x_2, x_3) = \begin{cases} 0 & (x_2, x_3) \in \text{R4} \\ \left[\frac{x_2 + x_3 - 2X_A + \frac{4}{n}X_B}{\frac{2}{n}X_B}\right] \frac{X_A - \frac{2}{n}X_B}{\frac{4}{n}X_B - X_A} & (x_2, x_3) \in \text{R5} \\ \frac{x_2}{\frac{2}{n}X_B} & (x_2, x_3) \in \text{R2} \\ \frac{x_3}{\frac{2}{n}X_B} & (x_2, x_3) \in \text{R3} \\ \frac{x_2 + x_3}{\frac{2}{n}X_B} - 1 & (x_2, x_3) \in \text{R1} \end{cases}$$

The univariate marginal distributions are given by $F\left(x_2, \frac{2}{n}X_B\right) = \frac{x_2}{\frac{2}{n}X_B}$ and $F\left(\frac{2}{n}X_B, x_3\right) = \frac{x_3}{\frac{2}{n}X_B}$. Thus F provides the necessary univariate marginal distributions for battlefields 2 and 3.

If $\frac{2}{n} = \frac{X_A}{X_B}$, then player A randomizes on only 2 battlefields and the support of this bivariate distribution function F collapses to the line segment $\left(\frac{2}{n}X_B, 0\right)$ to $\left(0, \frac{2}{n}X_B\right)$, i.e. the support is $\left\{(x_1, x_2) \in \mathbb{R}_+^2 | x_1 + x_2 = X_A\right\}$.[10]

If $\frac{2}{n} < \frac{X_A}{X_B} < \frac{3}{n}$, then, from the support of the bivariate distribution function F, it follows that

$$G(z) = \begin{cases} \left(\frac{z-\left(X_A - \frac{2}{n}X_B\right)}{\frac{2}{n}X_B}\right)\left(\frac{nX_A}{X_B} - 2\right) & z \in \left[X_A - \frac{2}{n}X_B, X_A\right) \\ 1 & z \geq X_A \end{cases}$$

Since $x_1 \equiv X_A - x_2 - x_3$, we have that the univariate marginal distribution for battlefield 1 places an atom of size $3 - \frac{nX_A}{X_B}$ at 0 and randomizes continuously on $\left[0, \frac{2}{n}X_B\right]$ with the remaining mass, and that for all realizations of (x_1, x_2, x_3) $x_1 + x_2 + x_3 = X_A$ with probability 1. Equivalently, the combination of $x_1 = X_A - z$ with the bivariate distribution F for x_2 and x_3 defines a trivariate distribution function, F', with support that uniformly places mass $\frac{X_A}{\frac{2}{n}X_B} - 1$ on each of the two following line segments

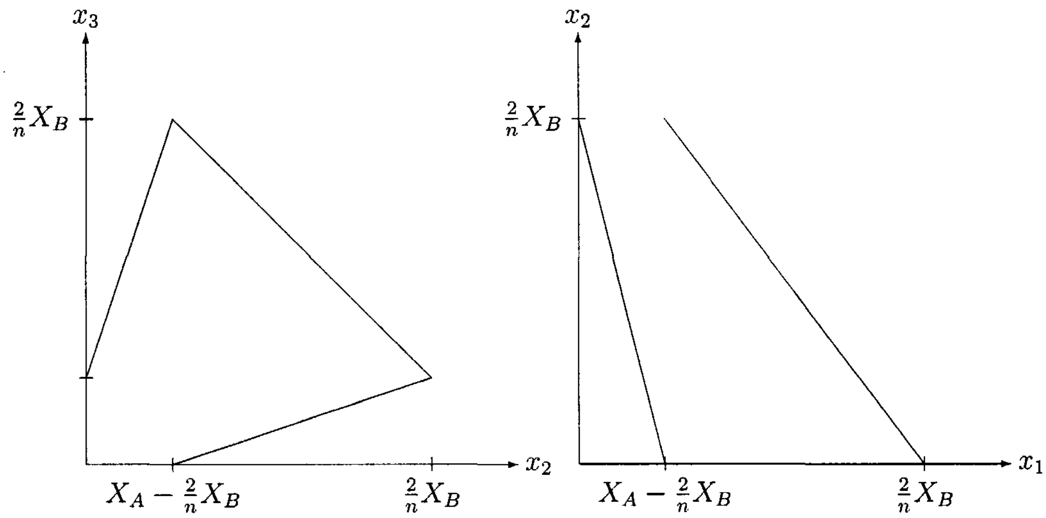
1. $\left(0, \frac{2}{n}X_B, X_A - \frac{2}{n}X_B\right)$ to $\left(\frac{2}{n}X_B, X_A - \frac{2}{n}X_B, 0\right)$
2. $\left(0, X_A - \frac{2}{n}X_B, \frac{2}{n}X_B\right)$ to $\left(\frac{2}{n}X_B, 0, X_A - \frac{2}{n}X_B\right)$

and uniformly places the remaining mass, $3 - \frac{nX_A}{X_B}$, on the line segment $\left(0, \frac{2}{n}X_B, X_A - \frac{2}{n}X_B\right)$ to $\left(0, X_A - \frac{2}{n}X_B, \frac{2}{n}X_B\right)$. The projection of this support onto the x_2, x_3-, x_1, x_3-, and x_1, x_2-planes is given in Figure 2.

[10]It should be pointed out that in the case that $\frac{2}{n} = \frac{X_A}{X_B}$, the bivariate distribution function F is exactly the Fréchet-Hoeffding lower bound 2-copula,

$$W = \max\left\{F(x_1) + F(x_2) - 1, 0\right\}$$

combined with $F(x_i) = \frac{x_i}{\frac{2}{n}X_B}$ for $x_i \in \left[0, \frac{2}{n}X_B\right]$ and i=1,2.



Projection of Support onto the x_2, x_3-plane

Projection of Support onto the x_1, x_2-plane (or x_1, x_3-plane)

Figure 2: Support of the Trivariate Distribution F'

If $\frac{X_A}{X_B} = \frac{3}{n}$, then player A randomizes on 3 battlefields according to the trivariate distribution function F' which has support that for $\frac{X_A}{X_B} = \frac{3}{n}$ uniformly places mass $\frac{1}{2}$ on each of the two following line segments

1. $\left(0, \frac{2}{n}X_B, X_A - \frac{2}{n}X_B\right)$ to $\left(\frac{2}{n}X_B, X_A - \frac{2}{n}X_B, 0\right)$
2. $\left(0, X_A - \frac{2}{n}X_B, \frac{2}{n}X_B\right)$ to $\left(\frac{2}{n}X_B, 0, X_A - \frac{2}{n}X_B\right)$

From the preceding discussion it is clear that each of the three univariate marginal distribution functions randomizes continuously on $\left[0, \frac{2}{n}X_B\right]$ and that for all realizations of (x_1, x_2, x_3) $x_1 + x_2 + x_3 = X_A$ with probability 1.

Similarly, for $\frac{3}{n} < \frac{X_A}{X_B} < \frac{4}{n}$ player A allocates force to at least three and not more than four battlefields. In this case, let $z = x_2 + x_3 + x_4$ and $x_1 = X_A - z$. Consider the support of the trivariate distribution function, F, for x_2, x_3, x_4 which uniformly places mass $2 - \frac{X_A}{\frac{2}{n}X_B}$ on each of the two following line segments

1. $\left(0, \frac{2}{n}X_B, X_A - \frac{2}{n}X_B\right)$ to $\left(\frac{2}{n}X_B, X_A - \frac{2}{n}X_B, 0\right)$
2. $\left(0, X_A - \frac{2}{n}X_B, \frac{2}{n}X_B\right)$ to $\left(\frac{2}{n}X_B, 0, X_A - \frac{2}{n}X_B\right)$

and uniformly places mass $\frac{X_A}{\frac{2}{n}X_B} - \frac{3}{2}$ on each of the two following line segments

1. $\left(0, 0, X_A - \frac{2}{n}X_B\right)$ to $\left(\frac{2}{n}X_B, X_A - \frac{2}{n}X_B, 0\right)$

2. $\left(0, X_A - \frac{2}{n}X_B, 0\right)$ to $\left(\frac{2}{n}X_B, 0, X_A - \frac{2}{n}X_B\right)$.

This support is shown in Figure 3.

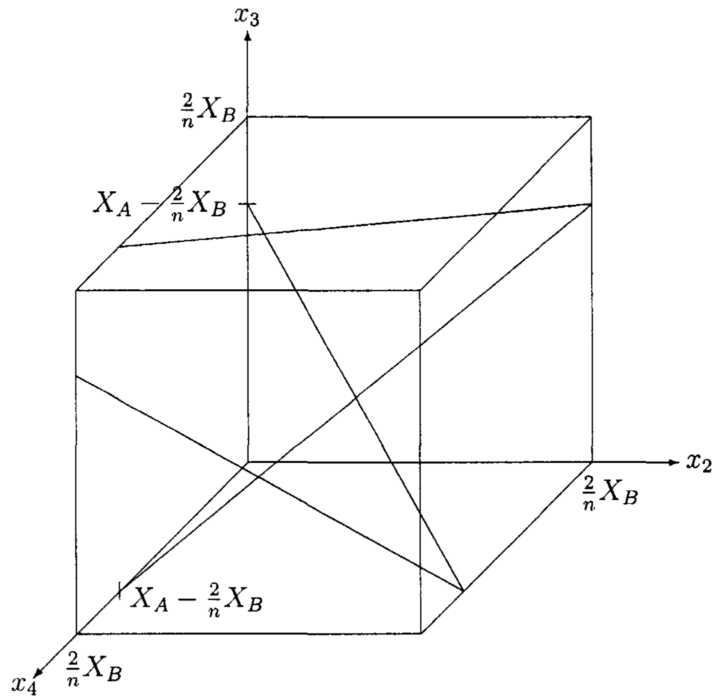


Figure 3: Support of the Trivariate Distribution F

Given this support, it is straightforward to establish that each of the three univariate marginal distribution functions randomizes continuously on $\left[0, \frac{2}{n}X_B\right]$. In addition, this trivariate distribution function has the property that the distribution of z places an atom of size $4 - \frac{nX_A}{X_B}$ at X_A and randomizes continuously on $\left[X_A - \frac{2}{n}X_B, X_A\right]$ with the remaining mass. Since at every point on the support $x_1 + x_2 + x_3 + x_4 = X_A$, it follows directly that the distribution on battlefield 1 places an atom of size $4 - \frac{nX_A}{X_B}$ at 0 and randomizes continuously on $\left[X_A - \frac{2}{n}X_B, X_A\right]$ with the remaining mass.

Since we can always use independent combinations of the bivariate and trivariate distributions used to establish that points 2 (b) and 3 (a) hold for $\frac{2}{n} \leq \frac{X_A}{X_B} \leq \frac{3}{n}$, the remaining cases, $\frac{4}{n} \leq \frac{X_A}{X_B} \leq 1$, follow directly. For example, in the case that $\frac{X_A}{X_B} = \frac{4}{n}$ player A can independently use the construction for $\frac{X_A}{X_B} = \frac{2}{n}$ twice. Thus, player A randomly selects $n-4$ battlefields which each receive zero force, breaks the remaining four battlefields into two sets of two battlefields, and independently uses the bivariate distribution function,

$$F(x,y) = \max\left\{\frac{x}{\frac{2}{n}X_B} + \frac{y}{\frac{2}{n}X_B} - 1, 0\right\}$$

for $x, y \in \left[0, \frac{2}{n}X_B\right]$ on each of the sets of two battlefields. Since these bivariate distribution functions are independent it is straightforward to show that the support across all four battlefields is contained in $\left\{\mathbf{x} \in \mathbb{R}^4_+ \middle| \sum_{i=1}^4 x_i = X_A\right\}$. In general, for all $\frac{4}{n} \leq \frac{X_A}{X_B} \leq 1$ there exist combinations of independent bi- and trivariate distribution functions to establish that points 2 (b) and 3 (a) hold for $\frac{4}{n} \leq \frac{X_A}{X_B} \leq 1$.

We now examine the case that $\frac{1}{n-1} \leq \frac{X_A}{X_B} < \frac{2}{n}$ as in Theorem 3. This proof is for player B. The existence of a sufficient n-variate distribution for player A in this parameter range is a special case of the Theorem 2 parameter range when $X_A = \frac{2}{n}X_B$. The construction of the n-variate distribution function is outlined as follows.

1. Player B randomly selects $\left\lfloor\frac{2X_B}{X_A}\right\rfloor - n$ of the battlefields and provides a force of X_A to those battlefields.

2. If $\left\lceil\frac{2X_B}{X_A}\right\rceil - \left\lfloor\frac{2X_B}{X_A}\right\rfloor = 1$, then:

   (a) Player B randomly selects $2n - \left\lceil\frac{2X_B}{X_A}\right\rceil$ of the remaining $2n - \left\lfloor\frac{2X_B}{X_A}\right\rfloor$ battlefields.

   (b) On the randomly selected $2n - \left\lceil\frac{2X_B}{X_A}\right\rceil$ battlefields, player B randomizes continuously on $[0, X_A]$ on each of these battlefields such that, letting z be the sum of player B's allocations of force on all $n-1$ of these battlefields and $G(z)$ be the distribution of z,

$$G(z) = 1 + \left(\frac{z - X_B}{X_A}\right)\left(1 - \frac{2X_B}{X_A} + \left\lfloor \frac{2X_B}{X_A} \right\rfloor\right)$$

for $z \in [X_B - X_A, X_B]$

(c) Defining the allocation of force on the remaining battlefield as $X_B - z$, it follows directly that the univariate distribution of force on the remaining battlefield places mass $\frac{2X_B}{X_A} - \left\lfloor \frac{2X_B}{X_A} \right\rfloor$ at X_A and randomizes continuously on $[0, X_A]$ with the remaining mass. In addition for all realizations, $\mathbf{x} \in \mathbb{R}^n_+$, of this strategy $\sum_{j=1}^{n} x^j = X_B$ with probability 1.

(d) There are $\binom{n}{2n-\left\lfloor \frac{2X_B}{X_A} \right\rfloor}\binom{2n-\left\lfloor \frac{2X_B}{X_A} \right\rfloor}{1}$ ways of dividing n battlefields into disjoint $\left(\left\lfloor \frac{2X_B}{X_A} \right\rfloor - n\right)$-subsets, $\left(2n - \left\lceil \frac{2X_B}{X_A} \right\rceil\right)$-subsets, and 1-subsets. Constructing an n-variate distribution as described above on each of the $\binom{n}{2n-\left\lfloor \frac{2X_B}{X_A} \right\rfloor}\binom{2n-\left\lfloor \frac{2X_B}{X_A} \right\rfloor}{1}$ possible divisions of the n battlefields into disjoint $\left(\left\lfloor \frac{2X_B}{X_A} \right\rfloor - n\right)$-subsets, $\left(2n - \left\lceil \frac{2X_B}{X_A} \right\rceil\right)$-subsets, and 1-subsets and weighting each of these n-variate distribution functions by $\left[\binom{n}{2n-\left\lfloor \frac{2X_B}{X_A} \right\rfloor}\binom{2n-\left\lfloor \frac{2X_B}{X_A} \right\rfloor}{1}\right]^{-1}$, the n-variate distribution function formed by taking the sum of these weighted n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\frac{\frac{2}{n}X_B}{X_A} - 1$ at X_A and randomize continuously on $[0, X_A]$.

3. If $\left\lceil \frac{2X_B}{X_A} \right\rceil - \left\lfloor \frac{2X_B}{X_A} \right\rfloor = 0$, then:

(a) On the remaining $2n - \frac{2X_B}{X_A}$ battlefields, player B randomizes continuously on $[0, X_A]$ on each of these battlefields such that, letting z be the sum of player B's allocation of force on all of the battlefields and $G(z)$ be the distribution of z,

$$G(z) = \begin{cases} 0 & z < X_B \\ 1 & z \geq X_B \end{cases}$$

Thus, for all realizations, $\mathbf{x} \in \mathbb{R}^n_+$, of this strategy $\sum_{j=1}^n x^j = X_B$ with probability 1.

(b) There are $\binom{n}{2n-\frac{2X_B}{X_A}}$ ways of dividing the n battlefields into disjoint $\left(\frac{2X_B}{X_A} - n\right)$-subsets and $\left(2n - \frac{2X_B}{X_A}\right)$-subsets. Constructing an n-variate distribution as described above on each of the $\binom{n}{2n-\frac{2X_B}{X_A}}$ possible divisions of the n battlefields into disjoint $\left(\frac{2X_B}{X_A} - n\right)$-subsets and $\left(2n - \frac{2X_B}{X_A}\right)$-subsets and weighting each of these n-variate distribution functions by $\left[\binom{n}{2n-\frac{2X_B}{X_A}}\right]^{-1}$, the n-variate distribution function formed by taking the sum of these weighted n-variate distribution functions has univariate marginal distribution functions which each have a mass point of $\left(\frac{\frac{2}{n}X_B}{X_A} - 1\right)$ at X_A and randomize continuously on $[0, X_A]$.

The pivotal steps in this construction are, again, points 2 (b) and 3 (a), and we will now show that there exist such multivariate distribution functions. In fact these multivariate distributions are quite similar to those used for the Theorem 2 parameter range. We will, thus, only provide the supports of the bivariate and trivariate distributions that establish that points 2 (b) and 3 (a) hold. Beginning with the case that $n-3 \le \frac{2X_B}{X_A} - n \le n-2$, from points 2 and 3 player B allocates a force of X_A to at least $n-3$ and not more than $n-2$ battlefields. Given that $n-3$ battlefields have received a force of X_A, for the three remaining battlefields let x_i denote the allocation of force to battlefield $i \in \{1,2,3\}$. Consider the support of a trivariate distribution function for x_1, x_2, x_3 which uniformly places mass $n-1-\frac{X_B}{X_A}$ on each of the two following line segments

1. $(0, X_A, X_B - X_A(n-2))$ to $(X_A, X_B - X_A(n-2), 0)$
2. $(0, X_B - X_A(n-2), X_A)$ to $(X_A, 0, X_B - X_A(n-2))$

and uniformly places the remaining mass, $\frac{2X_B}{X_A} - 2n + 3$, on the line segment $(X_A, 0, X_B - X_A(n-2))$ to $(X_A, X_B - X_A(n-2), 0)$. Given this support, it is straightforward to establish that the univariate marginal distribution functions on battlefields 2 and 3 randomize continuously on $\left[0, \frac{2}{n} X_B\right]$ and that the univariate marginal distribution function for battlefield 1 places an atom of size $\frac{2X_B}{X_A} - 2n + 3$ at X_A and randomizes continuously on $[0, X_A]$ with the remaining mass.

Similarly, for $n - 4 < \frac{2X_B}{X_A} - n < n - 3$ player B allocates a force of X_A to at least $n - 4$ and not more than $n - 3$ battlefields. Given that $n - 4$ battlefields have received a force of X_A, for the four remaining battlefields let x_i denote the allocation of force to battlefield $i \in \{1, 2, 3, 4\}$, $z' = x_2 + x_3 + x_4$ and $x_1 = X_A - z' - X_A (n - 4)$. Consider the support of a trivariate distribution function for x_2, x_3, x_4 which uniformly places mass $2 + \frac{X_B}{X_A} - n$ on each of the two following line segments

1. $(0, X_A, X_B - X_A (n - 2))$ to $(X_A, X_B - X_A (n - 2), 0)$
2. $(0, X_B - X_A (n - 2), X_A)$ to $(X_A, 0, X_B - X_A (n - 2))$

and uniformly places mass $n - \frac{X_B}{X_A} - \frac{3}{2}$ on each of the two following line segments

1. $(0, X_A, X_B - X_A (n - 2))$ to $(X_A, X_B - X_A (n - 2), X_A)$
2. $(0, X_B - X_A (n - 2), X_A)$ to $(X_A, X_A, X_B - X_A (n - 2))$

Given this support, it is straightforward to establish that each of the three univariate marginal distribution functions randomizes continuously on $[0, X_A]$. In addition, this trivariate distribution function has the property that the distribution of z' places an atom of size $4 + \frac{2X_B}{X_A} - 2n$ on $X_B - X_A (n - 3)$ and randomizes continuously on $[X_B - X_A (n - 3), X_B - X_A (n - 4)]$ with the remaining mass. Since at every point on the support $x_1 + x_2 + x_3 + x_4 = X_A$, it follows directly that the distribution on battlefield 1 places an atom of size $4 + \frac{2X_B}{X_A} - 2n$ at X_A and randomizes continuously on $[0, X_A]$ with the remaining mass.

Since we can always use independent combinations of the bivariate and trivariate distributions used to establish that points 2 (b) and 3 (a) hold for $n - 4 < \frac{2X_B}{X_A} - n \leq n - 2$, the remaining cases, $0 \leq \frac{2X_B}{X_A} - n \leq n - 4$, follow directly.

1.4 Conclusion

I conclude by noting the relationship between Borel's form of the Colonel Blotto game and Myerson's form of the Colonel Blotto game. Myerson (1993) presents a modified form of the Colonel Blotto game in which there are an infinite number of battlefields and the budget holds only in expectation (see Judd (1985) and Feldman and Gilles (1985) for a discussion of the application of the law of large numbers on a continuum). In recent years, this model has attracted interest, including redistributive politics applications such as: the incentives for generating budget deficits (Lizzeri (2002)), inefficiency of public good provision (Lizzeri and Persico (2001,2002)), and campaign spending regulation (Sahuguet and Persico (2004)). Myerson's justification for this simplified formulation is that:

> The hardest part of [the Colonel Blotto] problem was to construct joint distributions for allocations that always sum to the given total but give uniform marginal distributions for each battlefield/voter. I have avoided such difficulties here by allowing the offers to be made independently to the various voters and by only requiring that the budget constraint be satisfied in expected value. (p.858)

This paper demonstrates that the problem of constructing optimal joint distributions for Borel's form of the Colonel Blotto game can be separated into characterizing the set of univariate marginal distributions and establishing the existence of a mapping of this set into a joint distribution. In addition, this separation of the joint distribution into a set of univariate marginal distributions and an n-copula also highlights the connection between Borel's form of the Colonel Blotto game and Myerson's form of the Colonel Blotto game. In particular, for the players' levels of force specified in Theorems 2 and 3, in Borel's form of the Colonel Blotto game each player allocates all of their force with probability 1 and thus must allocate all of their force in expectation. It follows directly that the equilibrium distributions of

Myerson's form of the Colonel Blotto game correspond exactly with the equilibrium univariate marginal distributions in Borel's form of the Colonel Blotto game.

1.5 Bibliography

[1] Baye 96 Baye, M. R., D. Kovenock, and C. G. de Vries (1996), "The all-pay auction with complete information", *Economic Theory* 8:291-305.

[2] Borel, E. (1921), "La théorie du jeu les équations intégrales à noyau symétrique", *Comptes Rendus de l'Académie* 173:1304-1308; English translation by L. Savage (1953), "The theory of play and integral equations with skew symmetric kernels", *Econometrica* 21:97-100.

[3] Borel, E., and J. Ville (1938), *Application de la théorie des probabilitiés aux jeux de hasard*, Gauthier-Villars, Paris; reprinted in Borel E., and A. Chéron, *Théorie mathematique du bridge à la portée de tous*, Editions Jacques Gabay, Paris, 1991.

[4] Dasgupta, P., and E. Maskin (1986), "The existence of equilibrium in discontinuous economic games, I: theory," *Review of Economic Studies* 53:1-26.

[5] Feldman, M., and C. Gilles (1985), "An expository note on individual risk without aggregate uncertainty," *Journal of Economic Theory* 35:26-32.

[6] Gross, O., and R. Wagner (1950), "A continuous Colonel Blotto game", RM-408, RAND Corp., Santa Monica.

[7] Hillman, A. L., and J. G. Riley (1989), "Politically contestable rents and transfers," *Economics and Politics* 1:17-39.

[8] Judd, K. L. (1985), "The law of large numbers with a continuum of iid random variables," *Journal of Economic Theory* 35:19-25.

[9] Laslier, J. F. (2002), "How two-party competition treats minorities," *Review of Economic Design* 7:297-307.

[10] Laslier, J. F., and N. Picard (2002), "Distributive politics and electoral competition", *Journal of Economic Theory* 103:106-130.

[11] Lizzeri, A. (1999), "Budget deficits and redistributive politics," *Review of Economic Studies* 66:909-928.

[12] Lizzeri, A., and N. Persico (2001), "The provision of public goods under alternative electoral incentives," *American Economic Review* 91:225-239.

[13] Lizzeri, A., and N. Persico (2002), "The drawbacks of electoral competition," University of Pennsylvania, mimeo.

[14] Myerson, R. B. (1993), "Incentives to cultivate favored minorities under alternative electoral systems," *American Political Science Review* 87:856-869.

[15] Nelson, R. B. (1999), *An Introduction to Copulas*, Springer, New York.

[16] Sahuguet, N., and N. Persico (2004), "Campaign spending regulation in a model of redistributive politics", Forthcoming *Economic Theory.*

[17] Schweizer, B., and A. Sklar (1983), *Probabilistic Metric Spaces*, North-Holland, New York.

[18] Sklar, A. (1959), "Fonctions de répartition à n dimensions et leurs marges," *Publ. Inst. Statist. Univ. Paris* 8:229-231.

1.6 Appendix

The proof of Theorem 2, which is contained in the following lemmas, establishes that there exists a one-to-one correspondence between the equilibrium univariate marginal distributions of the Colonel Blotto game and the equilibrium distributions of bids from a unique set of two-bidder independent and identical simultaneous all-pay auctions. The uniqueness of the equilibrium univariate marginal distributions then follows from the characterization of the all-pay auction by Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996). In the discussion that follows, $\bar{s}_i^j$ and $\underline{s}_i^j$ are the upper and lower bounds of candidate i's distribution of force for battlefield j and $\frac{2}{n} \leq \frac{X_A}{X_B} \leq 1$.

Lemma 1: For each $i \in \{A, B\}$, $\lambda_i > 0$.

Proof: For Theorem 2's parameter range, in any equilibrium each player allocates all of their forces with probability 1. Thus, each player must also allocate all of their forces in expectation. Q.E.D.

These next four lemmas follow along the lines of the proofs in Baye, Kovenock, and de Vries (1996).

Lemma 2: For each $j \in \{1, \ldots, n\}$, $\bar{s}_{-i}^j = \bar{s}_i^j = \bar{s}^j$.

Lemma 3: In any equilibrium $\left\{F_i^j, F_{-i}^j\right\}_{j \in \{1,\ldots,n\}}$, no F_i^j can place an atom in the half open interval $(0, \bar{s}^j]$.

Lemma 4: For each $j \in \{1, \ldots, n\}$ and for each $i \in \{A, B\}$, $\frac{1}{n\lambda_i} F_{-i}^j(x) - x$ is constant $\forall\, x \in (0, \bar{s}^j]$.

Lemma 5: $\forall\, j \in \{1, \ldots, n\}$, $F_B^j(0) = 0$ and, thus, $\frac{1}{n\lambda_A} F_B^j(x) - x = 0$ $\forall$ $x \in [0, \bar{s}^j]$.

The following lemma characterizes the relationship between λ_A and λ_B.

Lemma 6: In equilibrium $\lambda_A = \lambda_B \frac{X_B}{X_A}$.

Proof: By way of contradiction, suppose that $\lambda_A \neq \lambda_B \frac{X_B}{X_A}$. For Theorem 2's parameter range, in any equilibrium each player allocates all of their forces with certainty and in expectation, thus

$$X_B \sum_{j=1}^{n} \int_0^{\bar{s}^j} x dF_A^j(x) = X_A \sum_{j=1}^{n} \int_0^{\bar{s}^j} x dF_B^j(x) \tag{1.3}$$

But, from lemmas 3, 4, and 5, it follows that

$$dF_A^j(x) = n\lambda_B dx \tag{1.4}$$

for all $x \in (0, \bar{s}^j]$, and

$$dF_B^j(x) = n\lambda_A dx \tag{1.5}$$

for all $x \in [0, \bar{s}^j]$. Substituting equations 4 and 5 into equation 3, we have

$$\lambda_B X_B \sum_{j=1}^{n} \int_0^{\bar{s}^j} nx dx = \lambda_A X_A \sum_{j=1}^{n} \int_0^{\bar{s}^j} nx dx$$

which is a contradiction since

$$\sum_{j=1}^{n} \int_0^{\bar{s}^j} nx dx = \sum_{j=1}^{n} \int_0^{\bar{s}^j} nx dx$$

but $\lambda_A \neq \lambda_B \frac{X_B}{X_A}$. Q.E.D.

The following lemma establishes the value of $\bar{s}^j$.

Lemma 7: $\bar{s}^j = \frac{1}{n\lambda_A}$.

Proof: From lemmas 4 and 5, we know that for each player i and any battlefield j

$$\frac{1}{n\lambda_i} F_{-i}^j(x) - x$$

is constant $\forall\ x \in (0, \bar{s}^j]$. It then follows that player i would never use a strategy that provides offers in $\left(\frac{1}{n\lambda_i}, \infty\right)$ since an offer of zero strictly dominates such a strategy. Noting that $\frac{1}{n\lambda_A} \leq \frac{1}{n\lambda_B}$, we have that $\bar{s}^j \leq \frac{1}{n\lambda_A}$ and that $\forall\ x \in (0, \bar{s}^j]$

$$\frac{1}{n\lambda_i} F_{-i}^j(x) - x \geq \frac{1}{n\lambda_i} - \bar{s}^j.$$

By way of contradiction, assume that $\bar{s}^j < \frac{1}{n\lambda_A}$ then by allocating a level of force to battlefield j that is greater than $\bar{s}^j$ by an arbitrarily small amount, player A can earn arbitrarily close to $\frac{1}{n\lambda_A} - \bar{s}^j > 0$ on battlefield j, which contradicts lemma 5. Q.E.D.

The following lemma establishes that there exists a unique pair λ_A, λ_B that satisfies the budget constraint.

Lemma 8: There exists a unique value for λ_A, and thus for λ_B. $\lambda_A = \frac{1}{2X_B}$ and thus $\lambda_B = \frac{X_A}{2X_B^2}$.

Proof: The budget constraint determines the unique pair λ_A, λ_B. Thus, λ_A solves

$$n \int_0^{\frac{1}{n\lambda_A}} xn\lambda_A dx = X_B$$

Solving for λ_A we have that

$$\lambda_A = \frac{1}{2X_B}.$$

It follows directly from lemma 6 that $\lambda_B = \frac{X_A}{2X_B^2}$. Q.E.D.

This completes the proof of Theorem 2.

2. Fiscal Federalism and the Incentives of Redistributive Politics

In the model of redistributive politics, political parties compete for representation in a legislature by simultaneously announcing binding commitments as to how they will allocate a budget across voters. Each voter votes for the party offering the highest level of utility, and each party's payoff is its representation in the legislature, which under proportional representation is equal to the fraction of votes received by that party. This model was originally formulated with a continuum of voters by Myerson (1993) and later with a finite population of voters by Laslier and Picard (2002).[1]

This paper extends Laslier and Picard's (2002) model of redistributive politics with a finite population of voters to allow for centralized and decentralized redistributive competition with local public goods and shows that this has important implications for both fiscal federalism and redistributive politics. In the centralized system, political parties compete for representation in the legislature by announcing binding commitments as to how they will allocate the aggregate budget to redistribution across the voters and to investment in the production of the local public goods in each of the jurisdictions. In the decentralized system, parties compete within each jurisdiction by announcing binding commitments as to how they will allocate that jurisdiction's budget to redistribution across the voters in that jurisdiction and to investment in the production of the local public good in that jurisdiction. In both

[1] In both of these games there are no pure strategy equilibrium. In Myerson (1993) an offer distribution is a probability distribution over $\mathbb{R}_+$ with the measure over each interval interpreted as the fraction of voters for whom the party's transfer has value in that interval. Since Myerson assumes a continuum of voters and offers that are independent across voters (each voter takes an independent draw from the offer distribution), one can appeal to Judd (1985) and Feldman and Gilles (1985) in assuming that the aggregate budget constraint holds with probability one and not just in expectation. In Laslier and Picard (2002) an offer distribution is an n-variate distribution over $\mathbb{R}^n_+$ with the property that the budget constraint holds with probability one.

systems, voters vote (sincerely) for the party that offers them the higher level of utility from both the transfer offered and the level of local public good provision.

In Laslier and Picard (2002), which follows from Borel's (1921) Colonel Blotto game, there are no pure strategy equilibrium. A mixed strategy is joint distribution function with one univariate marginal distribution for each voter, which represents the randomization of and the correlation between the transfers targeted at each voter. Similarly, there are no pure strategy equilibrium in the centralized and decentralized models of redistributive politics with local public goods. In both models, a mixed strategy is a joint distribution function with one dimension for each voter and one dimension for each jurisdiction, which represents the randomization and correlation of both the transfers targeted at each voter and the zero-one local public good provision decision for each jurisdiction. This paper demonstrates how to separate each party's best response correspondence into a set of univariate marginal distributions and a mapping from this set into a joint distribution.[2] I then completely characterize each parties' unique set of Nash equilibrium univariate marginal distributions for the centralized and decentralized models of redistributive politics with local public goods.[3]

In equilibrium, the level of inequality (as measured by the Gini-coefficient) arising from each party's redistributive/local public goods schedule is higher in a centralized system than in a decentralized system. In a centralized system the level of inequality arising in each party's redistributive/local public goods schedule is increasing in the utilities provided by the local public goods. Conversely, in a decentralized system, the level of inequality arising in each party's redistributive/local public goods schedule is decreasing in the utilities provided by the local public goods. In addition, if the utilities provided by the local public goods are above a minimal threshold, then

[2]See Nelson (1999) for an introduction to copulas, the mappings from univariate marginal distributions into joint distributions.

[3]While each player's set of equilibrium univariate marginal distributions of this game is unique, there are several distinct mappings (and thus several distinct joint distributions) of these sets of univariate marginal distributions into joint distributions with the property that the entire budget is spent with probability one.

centralization is also found to create greater inefficiencies in the provision of the local public good. However, the greater inefficiency of centralization is due to the incentives of redistributive competition with centralization and not to interjurisdictional externalities or heterogeneities in the production of or preferences for local public goods. In particular, centralization facilitates revenue sharing across jurisdictions. It is the combination of revenue sharing across jurisdictions and the targetablity of local public goods that lead to greater inefficiencies and greater inequality in a centralized system.

While closely related to the traditional theory of fiscal federalism (i.e. Musgrave (1959) and Oates (1972)), this paper departs from that theory in several important ways. First, this paper assumes that in a centralized system each party has the ability to choose the level of local public good provision for each of the jurisdictions rather than only a uniform level of local public good provision across all jurisdictions. Clearly, there are both theoretical and empirical justifications[4] for generalizing the local public good provision options in a model of a centralized system. Second, as is common in models of redistributive politics, the electorate is assumed to be homogeneous in both preferences for their local public good and in original endowment. In a centralized system, one possible goal of revenue sharing across jurisdictions is fiscal equalization, i.e. transfers from wealthy jurisdictions to poor jurisdictions. By assuming a homogeneous electorate, this paper highlights the strategic aspects of redistributive competition and the resulting inequality. Third, this paper assumes that there are no interjurisdictional externalities and no cost differentials in the production of the local public goods[5]. In this setting, the traditional theory of fiscal federalism provides no prescription for a centralized versus a decentralized system since the welfare under both systems is the same. In contrast, this paper highlights

[4]See for example Besley and Coate (2003) and Lockwood (2002), among others, who discuss this issue.

[5]In particular, in this paper the provision of the local public good has the same cost per voter in each jurisdiction.

important differences between the centralized and decentralized systems when the incentives of redistributive competition are taken into account.

Closely related is Lizzeri and Perisco's (2001) model of redistributive competition (over a continuum of voters) with public good provision. In that paper political parties compete for representation in a legislature by announcing binding commitments as to how they will allocate a budget across voters and investment in the production of a public good. As in this paper, the electorate is homogeneous in both their preferences for the public good and in original endowment, and each voter votes for the party offering the highest level of utility (from both redistribution and public good provision). That paper finds that in an electoral college system with winner-take-all in each jurisdiction and a continuum of jurisdictions each with a continuum of voters, the inefficiency of public good provision is much greater than in proportional system.

This paper generalizes Lizzeri and Perisco's (2001) public good production technology by allowing for local public good provision within each jurisdiction. It is this generalization that facilitates the comparison of centralized and decentralized redistributive competition with local public goods. In addition, in that paper there are a continuum of voters and the budget holds in expectation. This paper focuses on the case of a finite number of jurisdictions each with a finite number of voters and develops the necessary correlation structure to ensure that the budget holds with probability one. Lastly, Lizzeri and Persico (2001) focus solely on the inefficiency of public good provision. This paper also examines the inequality that results from centralized and decentralized redistributive competition with local public goods.

Section 2 presents the centralized and decentralized models of redistributive politics with local public good provision. Section 3 demonstrates how to separate the parties' best response correspondences into a set of univariate marginal distributions and a mapping from this set into a joint distribution; completely characterizes the unique set of Nash equilibrium univariate marginal distributions of the games of centralized and decentralized redistributive politics with local public goods; demonstrates the existence of a mapping from the set of equilibrium univariate marginal distributions into a joint distribution with the property that the budget is satisfied with probability one; and explores the nature of the inequality and inefficiency in the centralized and decentralized systems. Section 4 concludes.

2.1 The Model

Voters

The electorate consists of a finite number n of voters, which are denoted by $z \in \{1, \ldots, n\}$. Voters are partitioned among a finite number k of disjoint jurisdictions, where jurisdiction $j \in \{1, \ldots, k\}$ consists of a finite number $n_j \geq 3$ of voters such that $\sum_{j=1}^{k} n_j = n$. Voters are distinguished by the jurisdiction to which they belong, where voter z in jurisdiction j is denoted by $z(j)$. Each voter is endowed with 1 unit of a private homogeneous good.

In jurisdiction j the local public good provides a benefit of G_j to each of the voters in jurisdiction j. In each jurisdiction, j, the production of the local public good is a zero-one decision which is indicated by $\iota_i^j \in \{0, 1\}$ for party i. The production of the local public good in jurisdiction j requires all of district j's endowment, i.e. n_j units of the homogeneous good.[6] Throughout this paper we will focus on the case that production of the local public goods is efficient, i.e. $G_j \geq 1$.[7] Each voter in each jurisdiction receives an offer of a tax or transfer from each party. For voter z in jurisdiction j, let $t_i^{z(j)} \in \mathbb{R}_+$ denote that voter's amount of the private homogeneous good after party i's commitment of any taxes or transfers. Voters' utilities are additively separable in the private homogeneous good and the local public good. Thus, the utility that voter z in district j receives from party i who offers them $\left\langle \iota_i^j, t_i^{z(j)} \right\rangle$ is

$$u(i)_{z(j)} = t_i^{z(j)} + \iota_i^j G_j.$$

[6] As in Lizzeri and Persico (1998), this analysis is robust to the relaxation of this assumption.

[7] In the case that production of the local public goods is inefficient, the local public goods are not produced in any optimal strategy of the games of centralized and decentralized redistributive politics with local public goods, and the equilibrium becomes that of the model of redistributive politics without local public goods.

Each voter votes for the party that provides them with the higher level of utility. In the case that the parties provide the same level of utility to a voter, the parties win the voter with equal probability.

2.1.1 Centralized Political Competition

Parties

Two parties, A and B, make simultaneous offers of transfers to each of the n voters and production commitments to the local public good for each of the k jurisdictions. Each party's payoff is its vote share. The maximum tax that can be imposed upon a voter is equal to one unit of the private homogeneous good. Thus, each voter's allocation of the private homogeneous good, after any taxes or transfers, is nonnegative.

Centralized Strategies

As in both the model of pure redistributive politics and the Colonel Blotto game, there are no pure strategy equilibrium in either the centralized or decentralized redistributive politics with local public goods games. A mixed strategy for the game of centralized redistributive politics with local public goods, which we label a *centralized redistributive/local public goods schedule*, for party i is an $n+k$-variate distribution function $P_i : \{0,1\}^k \bigcup \mathbb{R}^n_+ \rightarrow [0,1]$. The $n+k$-tuple of the allocations of the private homogeneous good that result from party i's taxes/transfers to each of the n voters and production decisions for each of the k local public goods is a random $n+k$-tuple drawn from P_i with the set of univariate marginal distributions $\left\{\{L_i^j\}_{j=1}^k, \{F_i^z\}_{z=1}^n\right\}$. Since the production decision for each local public good is a zero-one decision, the k univariate marginal distribution functions, $\{L_i^j\}_{j=1}^k$, one univariate marginal distribution function for each district j, are each Bernoulli distributions. The probability that party i provides the local public good in district j, $E_{L_i^j}(x)$, is denoted by α_i^j. The remaining n univariate marginal distribution functions, $\{F_i^z\}_{z=1}^n$, one univariate marginal distribution function for each voter z, are the univariate marginal distributions of the allocations that result from party i's taxes/transfers to each voter z.

Each party's centralized redistributive/local public goods schedule must satisfy the aggregate budget constraint. The set of budget balancing centralized redistributive/local public good schedules is denoted by,

$$\mathfrak{B} = \left\{ \left\{ \{\iota^j\}_{j=1}^k, \{t^z\}_{z=1}^n \right\} \mid \sum_{j=1}^k \iota^j n_j + \sum_{z=1}^n t^z \leq n \right\}.$$

The support of any feasible centralized redistributive/local public goods schedule is contained in $\mathfrak{B}$.

Centralized Redistributive Politics with Local Public Goods

The *game of centralized redistributive politics with local public goods*, which we label

$$C\left\{ \{G_j, n_j\}_{j=1}^k \right\},$$

is the one-shot game in which parties compete by simultaneously announcing budget balancing centralized redistributive/local public goods schedules, each voter votes for the party that provides the higher utility, and candidates maximize their vote share.

2.1.2 Decentralized Political Competition

Parties

In each jurisdiction j, two parties, A and B, make simultaneous offers of either redistributive transfers to each of the voters in district j or provide the local public good in district j. Each party's payoff is its vote share. In the case that redistributive transfers are offered, the maximum tax that can be imposed on a voter is one unit of the private homogeneous good. Thus each voter's allocation of the private homogeneous good, after any taxes or transfers, is nonnegative.

Decentralized Strategies

A mixed strategy for the game of decentralized redistributive politics with local public goods in district j, which we label a *decentralized redistributive/local public goods schedule* in district j, for party i is an $n_j + 1$-variate distribution function, $H_i : \{0,1\} \cup \mathbb{R}_+^{n_j} \to [0,1]$. Let N_j denote the set of voters in jurisdiction j. The $n_j + 1$-tuple of the allocations of the private homogeneous good that result from party i's taxes/transfers to each of the n_j voters in jurisdiction j and the production decision for the local public good in jurisdiction j is a random $n_j + 1$-tuple drawn from H_i with the set of univariate marginal distributions $\left\{\{L_i\}, \{F_i^z\}_{z \in N_j}\right\}$. Since the production decision for the local public good is a zero-one decision, the single univariate marginal distribution function, $\{L_i\}$ is a Bernoulli distribution. The probability that party i provides the local public good, $E_{L_i}(x)$, is denoted by α_i. The remaining n_j univariate marginal distribution functions, $\{F_i^z\}_{z \in N_j}$, one univariate marginal distribution function for each voter in jurisdiction j, are the univariate marginal distributions of the allocations that result from party i's taxes/transfers to each voter z.

Each party's decentralized redistributive/local public goods schedule must satisfy the jurisdiction's budget constraint. The set of budget balancing decentralized redistributive/local public good schedules is denoted by,

$$\mathfrak{B}_j = \left\{ \left\{ \{\iota\}, \{t^z\}_{z \in N_j} \right\} | \iota n_j + \sum_{z \in N_j} t^z \leq n_j \right\}.$$

For each jurisdiction j, the support of any feasible decentralized redistributive/local public goods schedule is contained in $\mathfrak{B}_j$. One important distinction between the centralization and decentralization is that with decentralization the jurisdictional budget constraint requires that conditional on the decision to produce the local public good each voter in the district is taxed their entire endowment. That is for each $j \in \{1, \ldots, k\}$ and for all $z \in N_j$ $F_i^z(0|\iota_i = 1) = 1$ $i = A, B$.

Decentralized Redistributive Politics with Local Public Goods

The *game of decentralized redistributive politics with local public goods* for district j, which we label

$$D_j\{G_j, n_j\},$$

is the one-shot game in which parties compete by simultaneously announcing budget balancing decentralized redistributive/local public goods schedules, each voter votes for the party that provides the higher utility, and parties maximize their vote share.

2.2 Results

2.2.1 Optimal Univariate Marginal Distributions

To completely characterize the equilibrium univariate marginal distribution functions, we utilize n-copulas, the functions that map univariate marginal distributions into joint distributions.

Definition: Let I denote the unit interval $[0,1]$. An *n-copula* is a function C from I^n to I such that

1. For all $\mathbf{x} \in I^n$, $C(\mathbf{x}) = 0$ if at least one coordinate of $\mathbf{x}$ is 0, and if all coordinates of $\mathbf{x}$ are 1 except x_k, then $C(\mathbf{x}) = x_k$.
2. For every $\mathbf{x}$, $\mathbf{y} \in I^n$ such that $x_k \leq y_k$ for all $k \in \{1, \ldots, n\}$, the C-volume of the n-box $[x_1, y_1] \times \ldots \times [x_n, y_n]$,

$$V_C([\mathbf{x}, \mathbf{y}]) = \Delta_{x_n}^{y_n} \Delta_{x_{n-1}}^{y_{n-1}} \ldots \Delta_{x_2}^{y_2} \Delta_{x_1}^{y_1} C(\mathbf{t})$$

where

$$\begin{aligned} \Delta_{x_k}^{y_k} C(\mathbf{t}) &= C(t_1, \ldots, t_{k-1}, y_k, t_{k+1}, \ldots, t_n) \\ &\quad - C(t_1, \ldots, t_{k-1}, x_k, t_{k+1}, \ldots, t_n) \end{aligned}$$

is greater than or equal to 0.

Given the definition of an n-copula, we can state the crucial property of n-copulas that we will use.

Theorem 1 [Sklar's Theorem in n-dimensions]: Let H be an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$. Then there exists an n-copula C such that for all $\mathbf{x} \in \mathbb{R}^n$,

$$H(x_1, \ldots, x_n) = C(F_1(x_1), \ldots, F_n(x_n)) \tag{2.1}$$

Conversely, if C is an n-copula and $F_1, F_2, \ldots, F_n$ are univariate distribution functions, then the function H defined by equation 1 is an n-variate distribution function with univariate marginal distribution functions $F_1, F_2, \ldots, F_n$.

The proof of the two-dimensional version of Sklar's theorem is due to Sklar (1959). For a proof of the n-dimensional version see Schweizer and Sklar (1983).

One additional definition that will be used throughout the paper is the support of an n-variate distribution function.

Definition: The *support of an n-variate distribution function*, H, is the complement of the union of all open sets of $\mathbb{R}^n$ with H-volume zero.

Equilibrium in Centralized System

We now show that the in the game of centralized redistributive politics with local public goods the univariate marginal distribution functions and the $n+k$-copula are separate components of the parties' best response correspondences.

Proposition 1: In $C\left\{\{G_j, n_j\}_{j=1}^k\right\}$ each party's best response correspondence can be separated into the univariate marginal distribution functions and $n+k$-copula components.

Proof: In the game $C\left\{\{G_j, n_j\}_{j=1}^k\right\}$, for a given P_{-i} each party maximizes its expected vote share. There are two cases to consider. The first is that the upper bound of the support, $\bar{s}_i^z$, of each univariate marginal distribution function, F_i^z, is less than or equal to G_j, i.e. $\bar{s}_i^z \leq G_j \ \forall \ z$ and

i. The second is that the upper bound of the support, $\bar{s}_i^z$, of one or more of the univariate marginal distribution functions, F_i^z, is more than G_j, i.e. $\bar{s}_i^z > G_j$ for at least one z and i. The proof for each of the these cases follows similar lines, and given that Lemma 1 in the appendix establishes that in any optimal strategy $\bar{s}_i^z \leq G_j \ \forall \ z$ and i, we will focus on the first case. Thus we have that party i's optimization problem can be written as

$$\max_{P_i} \tfrac{1}{n} \textstyle\sum_{j=1}^{k} n_j \left(1 - \alpha_{-i}^j\right) \alpha_i^j +$$
$$\tfrac{1}{n} \textstyle\sum_{j=1}^{k} \left(1 - \alpha_{-i}^j - \alpha_i^j + 2\alpha_{-i}^j \alpha_i^j\right) \sum_{z \in N_j} \int_0^{\infty} F_{-i}^{z(j)}(x)\, dF_i^{z(j)}$$

subject to the constraint that the support of P_i is contained in $\mathfrak{B}$ or equivalently the P_i-volume over the region

$$\left\{ \left\{ \iota^j, \left\{ t^{z(j)} \right\}_{z \in N_j} \right\}_{j=1}^{k} \mid \sum_{j=1}^{k} \left(\iota^j n_j + \sum_{z \in N_j} t^{z(j)} \right) > n \right\}$$

is 0.

In any optimal strategy $Pr_{P_i}\left[\sum_{j=1}^{k} \left(\iota_i^j n_j + \sum_{z \in N_j} t_i^{z(j)}\right) = n\right] = 1$ since each party i's centralized redistributive/local public goods schedule must have support in $\mathfrak{B}$, and each party will allocate all of the budget. Thus, the P_i-volume over the region

$$\left\{ \left\{ \iota^j, \left\{ t^{z(j)} \right\}_{z \in N_j} \right\}_{j=1}^{k} \mid \sum_{j=1}^{k} \left(\iota^j n_j + \sum_{z \in N_j} t^{z(j)} \right) < n \right\}$$

is 0 and

$$E_{P_i}\left(\sum_{j=1}^{k} \left(\iota^j n_j + \sum_{z \in N_j} t^{z(j)} \right) \right) = n.$$

Noting that

$$E_{P_i}\left(\textstyle\sum_{j=1}^{k} \left(\iota^j n_j + \sum_{z \in N_j} t^{z(j)}\right)\right) =$$
$$\textstyle\sum_{j=1}^{k} \left(n_j E_{L_i^j}\left(y_i^j\right) + \sum_{z \in N_j} E_{F_i^{z(j)}}\left(x_i^{z(j)}\right)\right)$$

and $E_{L_i^j}\left(y_i^j\right) = \alpha_i^j$, we have that

$$\sum_{j=1}^{k} \left(\alpha_i^j n_j + \sum_{z \in N_j} E_{F_i^{z(j)}}\left(x_i^{z(j)}\right) \right) = n \qquad (2.2)$$

That is if the budget holds with certainty, it must also hold in expectation.

Then from Theorem 1 the $n+k$-variate distribution P_i is equivalent to the set $\left\{L_i^j, \left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k$ combined with an appropriate $n+k$-copula, C. Thus, from equation 2 and the fact that the P_i-volume over the regions

$$\left\{\left\{\iota^j, \{t^{z(j)}\}_{z\in N_j}\right\}_{j=1}^k \mid \sum_{j=1}^k \left(\iota^j n_j + \sum_{z\in N_j} t^{z(j)}\right) > n\right\}$$

and

$$\left\{\left\{\iota^j, \{t^{z(j)}\}_{z\in N_j}\right\}_{j=1}^k \mid \sum_{j=1}^k \left(\iota^j n_j + \sum_{z\in N_j} t^{z(j)}\right) < n\right\}$$

is 0, the Lagrangian of party i's optimization problem can be written as

$$\max_{\left\{L_i^j,\left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k} \frac{1}{n}\sum_{j=1}^k n_j\left(1-\alpha_{-i}^j-\lambda_i\right)\alpha_i^j +$$

$$\frac{1}{n}\sum_{j=1}^k \sum_{z\in N_j}\left[\lambda_i \int_0^\infty \left[\frac{1-\alpha_{-i}^j-\alpha_i^j+2\alpha_{-i}^j\alpha_i^j}{\lambda_i} F_{-i}^{z(j)}(x) - x\right] dF_i^{z(j)}\right] + \lambda_i$$

where the set of univariate marginal distribution functions

$$\left\{L_i^j, \left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k$$

satisfy the constraint that there exists a $n+k$-copula, C, such that the support of the $n+k$-variate distribution

$$C\left(\left\{L_i^j, \left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k\right)$$

is contained in

$$\left\{\left\{\iota^j, \{t^{z(j)}\}_{z\in N_j}\right\}_{j=1}^k \mid \sum_{j=1}^k \left(\iota^j n_j + \sum_{z\in N_j} t^{z(j)}\right) = n\right\}.$$

The proof of the second case follows directly. Q.E.D.

Note that from Theorem 1 an n-variate distribution is equivalent to a set of univariate marginal distribution functions, $\{F^i\}_{i=1}^n$, and an n-copula, C. This in combination with the payoff function of this class of games allows us to separate the parties' best response correspondences into the univariate marginal distribution and $n+k$-copula components.

The following Theorem completely characterizes the set of equilibrium univariate marginal distributions for the game of centralized redistributive politics with local public goods. Section 3.2 establishes the existence of an $n+k$-variate copula with the necessary property that the combination of the $n+k$-copula and the set of equilibrium univariate marginal distributions allocate the aggregate budget with probability one.

Theorem 2: The unique Nash equilibrium univariate marginal distributions of the game $C\left\{\{G_j, n_j\}_{j=1}^{k}\right\}$ are for each party to produce the local public goods and offer transfers according to the following univariate distribution functions. For each party i and jurisdiction j

$$\forall\ z \in N_j \qquad F_i^{z(j)}(x) = x \qquad x \in [0,1]$$

$$L_i^j(y) = \begin{cases} \frac{1}{2} & y = 0 \\ 1 & y = 1 \end{cases}$$

The expected payoff for each party is $\frac{1}{2}$ of the vote share.

Proof: We begin by showing that this is an equilibrium. First, in any optimal strategy the budget holds with certainty and thus in expectation. Assuming that there exists a sufficient $n+k$-copula (which is established in section 3.2), this is a feasible strategy since:

$$\sum_{j=1}^{k}\left(\alpha_i^j n_j + \sum_{z\in N_j} E_{F_i^{z(j)}}\left(x_i^{z(j)}\right)\right) = n$$

Then given that party B is following the equilibrium strategy, it is never a best response for party A to provide transfers outside the support of party B's transfers. Thus we have that the payoff to party A when it chooses to provide transfers according to an arbitrary strategy $\left\{G_i^j, \left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^{k}$ is:

$$\tfrac{1}{2n}\textstyle\sum_{j=1}^{k} n_j\alpha_i^j + \tfrac{1}{2n}\sum_{j=1}^{k}\sum_{z\in N_j}\int_0^1 x dF_i^{z(j)}$$

But from equation (2) of Proposition 1 it follows that

$$\frac{1}{2n}\sum_{j=1}^{k} n_j \alpha_i^j + \frac{1}{2n}\sum_{j=1}^{k}\sum_{z\in N_j}\int_0^1 x dF_i^{z(j)} \leq \frac{1}{2}$$

which holds with equality if and only if party A uses a strategy that spends the entire budget in expectation, as does the equilibrium strategy. Thus party A's vote share cannot be increased by deviating to another strategy. The argument for party B is symmetric.

The proof of uniqueness is contained in the appendix. Q.E.D.

This result holds regardless of the benefit of the local public good in each district.

Equilibrium in a Decentralized System

A similar result applies to the game of decentralized redistributive politics with local public goods.

Proposition 2: In $D_j\{G_j, n_j\}$ each party's best response correspondence can be separated into the univariate marginal distribution functions and $n_j + 1$-copula components.

Proof: In the game $D_j\{G_j, n_j\}$, for a given H_{-i} each party maximizes its expected vote share in jurisdiction j. Thus, in jurisdiction j the optimization problem for party i can be written as

$$\max_{H_i} \tfrac{1}{n_j}(1-\alpha_i)\,\alpha_{-i}\textstyle\sum_{z\in N_j}\left(1 - F_i^z(G_j|\iota_i = 0)\right) +$$
$$\tfrac{1}{n_j}(1-\alpha_{-i})(1-\alpha_i)\textstyle\sum_{z\in N_j}\int_0^\infty F_{-i}^z(x|\iota_{-i}=0)\,dF_i^z(x|\iota_i=0) +$$
$$\tfrac{1}{n_j}(1-\alpha_{-i})\,\alpha_i\textstyle\sum_{z\in N_j} F_{-i}^z(G_j|\iota_{-i}=0) + \tfrac{\alpha_i\alpha_{-i}}{2}$$

subject to the constraint that the support of H_i is contained in $\mathfrak{B}_j$ or equivalently the H_i-volume over the region

$$\left\{\left\{\{\iota\},\{t^z\}_{z\in N_j}\right\}|\iota n_j + \sum_{z\in N_j} t^z > n_j\right\}$$

is 0.

In any optimal strategy $Pr_{H_i|\iota_i=0}\left[\sum_{z\in N_j} t_i^z = n_j\right] = 1$ since each party i's decentralized redistributive/local public good provision schedule must have support in $\mathfrak{B}_j$, and each party will allocate all of jurisdiction j's budget. Thus, the H_i-volume over the region

$$\left\{\left\{\{\iota\},\{t^z\}_{z\in N_j}\right\}|\iota n_j + \sum_{z\in N_j} t^z < n_j\right\}$$

is 0 and

$$E_{H_i|\iota_i=0}\left(\sum_{z\in N_j} t^z\right) = n_j.$$

That is if the budget holds with certainty it must hold in expectation. Noting that

$$E_{H_i|\iota_i=0}\left(\sum_{z\in N_j} t^z\right) = \sum_{z\in N_j} E_{F_i^z|\iota_i=0}\left(x_i^z\right)$$

we have that

$$\sum_{z\in N_j} E_{F_i^z|\iota_i=0}\left(x_i^z\right) = n_j \tag{2.3}$$

From Theorem 1 the n_j+1-variate distribution H_i is equivalent to the set $\left\{\{L_i\},\{F_i^z\}_{z\in N_j}\right\}$ combined with an appropriate n_j+1-copula, C. Thus, the Lagrangian of party i's optimization problem can be written as

$$\begin{array}{c} \max_{\left\{L_i,\{F_i^z\}_{z\in N_j}\right\}} \frac{1}{n_j}\left(1-\alpha_i\right)\alpha_{-i}\sum_{z\in N_j}\left(1-F_i^z\left(G_j|\iota_i=0\right)\right)+ \\ \frac{\lambda_i}{n_j}\sum_{z\in N_j}\int_0^\infty\left(\frac{(1-\alpha_i)(1-\alpha_{-i})}{\lambda_i}F_{-i}^z\left(x|\iota_{-i}=0\right)-x\right)dF_i^z\left(x|\iota_i=0\right)+ \\ \frac{1}{n_j}\left(1-\alpha_{-i}\right)\alpha_i\sum_{z\in N_j}F_{-i}^z\left(G_j|\iota_{-i}=0\right)+\frac{\alpha_i\alpha_{-i}}{2}+\lambda_i \end{array}.$$

where the set of univariate marginal distribution functions

$$\left\{\{L_i\},\{F_i^z\}_{z\in N_j}\right\}$$

satisfy the constraint that there exists a n_j+1-copula, C, such that the support of the n_j+1-variate distribution $C\left(L_i,\{F_i^z\}_{z\in N_j}\right)$ is contained in

$$\left\{\left\{\{\iota\},\{t^z\}_{z\in N_j}\right\}|\iota n_j + \sum_{z\in N_j} t^z = n_j\right\}.$$

Q.E.D.

Given the fact that if the budget must hold with certainty then it must hold in expectation, it follows directly that the set of Nash equilibrium univariate marginal distributions of the game of decentralized redistributive politics with local public goods coincide exactly with the equilibrium of Lizzeri and Persico's (2001) model of redistributive politics with public good provision. Thus for the game of decentralized redistributive politics with local public goods we have the following.

Theorem 3: (Lizzeri and Persico (2001)) The unique Nash equilibrium univariate marginal distribution functions of the game $D_j\{G_j, n_j\}$ are for each party to produce the local public goods and offer transfers according to the following univariate distribution functions. If $1 \leq G_j \leq 2$ in district j, then for party i

$$L_i(y) = \begin{cases} 2 - G_j & y = 0 \\ 1 & y = 1 \end{cases}$$

and $\forall\ z \in N_j$

$$F_i^{z(j)}(x|\iota_i = 0) = \begin{cases} \frac{1}{2}\left(\frac{x}{2-G_j}\right) & 0 \leq x < 2 - G_j \\ \frac{1}{2} & 2 - G_j \leq x < G_j \\ \frac{1}{2}\left(1 + \frac{x-G_j}{2-G_j}\right) & G_j \leq x < 2 \\ 1 & x \geq 2 \end{cases}$$

If $G_j > 2$ then both parties provide the local public good certainty. In both cases, the expected payoff for each party is $\frac{1}{2}$ of the vote share.

Given Proposition 2, the proof of existence and uniqueness of this equilibrium is a straightforward extension of Lizzeri and Persico's (2001) result for a continuum of voters and is thus omitted.

2.2.2 Existence of Sufficient n-copulas

Subject to the constraint that there exist sufficient n-copulas, Theorems 2 and 3 characterize the unique sets of the games of centralized and decentralized redistributive politics with local public goods. There are no known existence results for

$n+k$-copulas or n_j+1-copulas with our necessary properties. However from Theorem 1, it is sufficient to show that there exists at least one $n+k$-variate distribution and one n_j+1-variate distribution with the necessary properties. To avoid non-generic cases which complicate the statement of the proofs and results, we will focus on the case in which n, k, and, for all $j \in \{1, \ldots, k\}$, n_j are even. In addition, it will be assumed that each jurisdiction has the same number of voters, i.e. $n_j = n_{-j}$ $\forall$ $j, -j \in \{1, \ldots, k\}$.[8]

Centralized System

Since $n \geq 8$,[9] there are several graphical solutions for generating n-variate distributions over the set $\mathfrak{B}$.[10] In the discussion that follows, we will focus on the classic 'disk' solution due to Borel (Borel and Ville (1938)) and the generalized 'disk' solution, due to Gross and Wagner (1950).

Since we have assumed that k is even and all jurisdictions are of the same size, we can simplify the construction of a sufficient $n+k$-copula by separately constructing an n-copula and a k-copula which are independent. We begin with the n-variate marginal distribution function of P_i with the set of univariate marginal distribution functions $\{F_i^z\}_{z=1}^n$. Consider a regular n-gon with sides, $z \in \{1, \ldots, n\}$, of length $\tan\left(\frac{\pi}{n}\right)$. Let Ω be the center of this regular n-gon. The diameter of the circle inscribed in this n-gon with center Ω is 1.[11] Let S be the hemisphere of diameter 1 centered at Ω. Let M be a point randomly chosen from the surface of S, according to the uniform distribution on the surface of S. Let M' be the projection of M on the plane that contains the n-gon. Finally, let t^z be the distance from M' to the side z of the n-gon.

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[8]The main results of this paper hold for all feasible partitions of the electorate, but the construction of sufficient n-copulas is remarkably more cumbersome in the case that the jurisdictions are of different sizes.

[9]$n \geq 8$ since $k \geq 2$ and $n_j \geq 4$ for $j \in \{1, \ldots, k\}$.

[10]See Gross and Wagner (1950).

[11]For a review of the properties of regular n-gons see Harris and Stocker (1998).

For the k-variate marginal distribution function of P_i with the set of univariate marginal distribution functions $\{L_i^j\}_{j=1}^k$, partition the jurisdictions into two groups of equal size and randomly choose one of the groups for which the local public good is provided in every jurisdiction. Since $n_j = n_{-j}$ for all j and k is even it follows directly that such a partition exists. Given this we have the following.

Proposition 3: Let P_i^* denote the $n+k$-variate distribution function over $\left\{\left\{\iota^j, \{t^{z(j)}\}_{z\in N_j}\right\}_{j=1}^k \mid \sum_{j=1}^k \left(\iota^j n_j + \sum_{z\in N_j} t^{z(j)}\right) = n\right\}$ induced by the construction outlined above. Then, P_i^* generates the equilibrium univariate marginal distributions in Theorem 2 and a sufficient $n+k$-copula.

The construction of the n-variate distribution function for the randomization of the transfers is the standard construction of the 'disk' solution (for a proof see Laslier (2002), Laslier and Picard (2002), or Gross and Wagner (1950)). The independent combination of this n-variate distribution with the construction of the k-variate distribution given above defines an $n+k$-variate distribution function with support on the set $\left\{\left\{\iota^j, \{t^{z(j)}\}_{z\in N_j}\right\}_{j=1}^k \mid \sum_{j=1}^k \left(\iota^j n_j + \sum_{z\in N_j} t^{z(j)}\right) = n\right\}$ that generates the equilibrium marginal distributions in Theorem 2. Thus a sufficient $n+k$-copula exists.

Decentralized System

Since $n_j \geq 4$ and even, we can use the 'disk' solution to generate the candidates' distributions of redistributive transfers. For the n_j-variate marginal distribution of H_i conditional on $\iota_i = 0$ with the set of univariate marginal distributions $\{F_i^Z|\iota_i = 0\}_{z\in N_j}$, consider a regular n_j-gon with sides, $z \in \{1, \ldots, n_j\}$, of length $(2 - G_j)\tan\left(\frac{\pi}{n}\right)$. Let Ω be the center of this regular n_j-gon. The diameter of the circle inscribed in this n-gon with center Ω is $2 - G_j$. Let S be the hemisphere of diameter $2 - G_j$ centered at Ω. Let M be a point randomly chosen from the surface of S, according to the uniform distribution on the surface of S. Let M' be the projection of M on the plane that contains the n-gon. Finally, let t^z be the distance from M' to the side z of the n-gon. Then randomly assign an additional transfer of G_j to half of the voters in district j.

Combining this construction for the distribution of transfers conditional on not providing the local public good with the equilibrium probability of providing the local public good we have the following.

Proposition 4: Let H_i^* denote the n_j+1-variate distribution function over $\left\{\left\{\iota, \left\{t^{z(j)}\right\}_{z\in N_j}\right\} | \iota n_j + \sum_{z\in N_j} t^{z(j)} = n_j\right\}$ induced by the construction outlined above. Then, H_i^* generates the equilibrium univariate marginal distributions in Theorem 3 and a sufficient n_j+1-copula.

The construction of H_i^* is a straightforward extension of the standard construction of the 'disk' solution (for a proof see Laslier (2002), Laslier and Picard (2002), or Gross and Wagner (1950)).

2.2.3 Centralization vs. Decentralization

We now apply the equilibrium characterizations of the centralized and decentralized systems to compare the inequality and inefficiency that arise in each. Corollary 1 examines the equilibrium level of inequality arising in the centralized system and corollary 2 examines the equilibrium level of inequality arising in the decentralized system. In the centralized system, the Lorenz curves for the parties' centralized redistributive/local public goods schedules are piecewise quadratic functions that depend critically on the utilities provided by each of the local public goods. To simplify this analysis, we will focus on the case that the utilities provided by the local public goods are the same for each jurisdiction, i.e. $G_j = G \; \forall \; j \in \{1, \ldots, k\}$.

Corollary 1: For each party $i = A, B$, the inequality (as measured by the Gini-coefficient of inequality) arising from party i's equilibrium centralized redistributive/local public goods schedule is increasing in the utility provided by the local public good, G, for all efficient levels of the utility provided by the local public good. More precisely, the Gini-coefficient of party i's equilibrium centralized redistributive/local public goods schedule is $C_i^C = 1 - \frac{G}{2(1+G)} - \frac{5}{6(1+G)}$.

Proof: From Theorem 2 and proposition 3, the proportion of voters who receive a utility level from party i's equilibrium centralized redistributive/local public goods schedule that is less than or equal to x is

$$\tilde{F}_i(x) = \begin{cases} \frac{x}{2} & x \in [0,1) \\ \frac{1}{2} & x \in [1,G) \\ \frac{1}{2} + \frac{x-G}{2} & x \in [G, G+1] \end{cases}$$

By definition the Lorenz curve for $\tilde{F}_i$ is

$$L_i(y) = \frac{\int_0^y \tilde{F}_i^{-1}(x)dx}{\int_0^1 \tilde{F}_i^{-1}(x)dx}, \quad y \in [0,1],$$

which is equivalent to

$$L_i(y) = \begin{cases} \frac{2y^2}{1+G} & y \in \left[0, \frac{1}{2}\right) \\ \frac{\frac{1}{2} + 2G\left(y - \frac{1}{2}\right) + 2\left(y - \frac{1}{2}\right)^2}{1+G} & y \in \left[\frac{1}{2}, 1\right] \end{cases}$$

By definition, the Gini-coefficient for $\tilde{F}_i$ is

$$C_i^C(G) = 1 - 2\int_0^1 L_i(x)\,dx.$$

Simplifying we have $C_i^C = 1 - \frac{G}{2(1+G)} - \frac{5}{6(1+G)}$. It follows that $\frac{\partial C_i^C}{\partial G} > 0$. Q.E.D.

Similarly, in the decentralized case we have the following.

Corollary 2: For each party $i = A, B$ and each jurisdiction $j \in \{1, \ldots, k\}$, the inequality (as measured by the Gini-coefficient of inequality) arising from party i's equilibrium decentralized redistributive/local public goods schedule in jurisdiction j is decreasing in the utility provided by the local public good, G_j, for $1 \leq G_j \leq 2$. If $G_j > 2$ then the local public good is provided with certainty and there is no inequality. More precisely, for

$1 \leq G_j \leq 2$ the Gini-coefficient of party i's equilibrium decentralized redistributive/local public goods schedule is $C_i^D = \frac{1}{6}\left(2 + G_j - G_j^2\right)$

Proof: From Theorem 3 and proposition 4, conditional on party i choosing to not provide the local public good in jurisdiction j the proportion of voters in jurisdiction j who receive a utility level from party i's equilibrium centralized redistributive/local public goods schedule that is less than or equal to x is

$$\tilde{F}_{i,j}\left(x|\iota_i = 0\right) = \begin{cases} \frac{x}{2(2-G_j)} & x \in [0, 2-G_j) \\ \frac{1}{2} & x \in [2-G_j, G_j) \\ \frac{1}{2} + \frac{x-G_j}{2(2-G_j)} & x \in [G_j, 2] \end{cases}$$

By definition the Lorenz curve for $\tilde{F}_{i,j}|\iota_i = 0$ is

$$L_i\left(y\right) = \frac{\int_0^y \tilde{F}_{i,j}^{-1}(x\iota_i=0)dx}{\int_0^1 \tilde{F}_{i,j}^{-1}(x\iota_i=0)dx}, \quad y \in [0,1],$$

which is equivalent to

$$L_i\left(y\right) = \begin{cases} \left(2-G_j\right)y^2 & y \in \left[0, \frac{1}{2}\right) \\ \frac{2-G_j}{4} + G_j\left(y - \frac{1}{2}\right) + \left(2 - G_j\right)\left(y - \frac{1}{2}\right)^2 & y \in \left[\frac{1}{2}, 1\right] \end{cases}$$

By definition, the Gini-coefficient for $\tilde{F}_{i,j}|\iota_i = 0$ is

$$C_i^D\left(G_j|\iota_i = 0\right) = 1 - 2\int_0^1 L_i\left(x\right)dx.$$

Simplifying we have $C_i^D|\iota_i = 0 = \frac{1}{6} + \frac{G_j}{6}$. Then note that party i offers the local public good with probability $G_j - 1$, and that when the public good is offered there is no inequality. It follows that the unconditional Gini-coefficient is given by

$$C_i^D = \frac{1}{6}\left(2 + G_j - G_j^2\right)$$

and that $\frac{\partial C_i^D}{\partial G_j} < 0$. Q.E.D.

Note that at the point where production of the local public goods becomes efficient, $G = 1$, the inequalities in the centralized and decentralized systems are the same. In addition, the inequality in the centralized system is increasing in the utility provided by the local public goods, G, while the inequality in the decentralized system is decreasing in the utility provided by the local public good, G_j. Thus, for all strongly efficient levels of utility provided by the local public goods, $G > 1$, the inequality arising from the centralized system is greater than that in the decentralized system.

Corollary 3: For all strongly efficient levels of utility provided by the local public goods, $G > 1$, the inequality arising from the centralized system is greater than that arising from the decentralized system.

In both the centralized and decentralized systems there is ex ante inefficiency in the equilibrium outcomes. However, from Theorem 2 each local public good is produced with probability $\frac{1}{2}$ in the centralized system, while from Theorem 3 each local public good is produced with probability $G_j - 1$ in the decentralized system. It follows that the ex ante utility is higher in the centralized system if $1 \leq G < \frac{3}{2}$ and higher in the decentralized system if $G_j > \frac{3}{2}$.

Corollary 4: The ex ante utility is higher in the centralized system if $1 \leq G < \frac{3}{2}$ and higher in the decentralized system if $G_j > \frac{3}{2}$.

Thus, once the utilities of the local public goods are above a minimal threshold then the decentralized system is more efficient than the decentralized system.

2.3 Conclusion

This paper extends Laslier and Picard's (2002) model of redistributive politics with a finite population of voters to allow for centralized and decentralized redistributive competition with local public goods and shows that this highlights important distinctions between centralization and decentralization that are absent from the traditional theory of fiscal federalism. In equilibrium, the level of inequality (as measured by the Gini-coefficient) arising from each party's redistributive/local public goods schedule is higher in a centralized system than in a decentralized system. In a centralized system the level of inequality arising in each party's redistributive/local public goods schedule is increasing in the utilities provided by the local public goods. Conversely, in a decentralized system, the level of inequality arising in each party's redistributive/local public goods schedule is decreasing in the utilities provided by the local public goods. In addition, if the utilities provided by the local public goods are above a minimal threshold, then centralization is also found to create greater inefficiencies in the provision of the local public good. However, the greater inefficiency and inequality of centralization is due to the targetability of local public goods and the ability to share revenue across districts and not to interjurisdictional externalities or heterogeneities in the production of or preferences for local public goods.

2.4 Bibliography

[1] Baye, M. R., D. Kovenock, and C. G. de Vries (1996), "The all-pay auction with complete information", *Economic Theory* 8:291-305.

[2] Besley, T., and S. Coate (2003), "Centralized versus decentralized provision of local public goods: a political economy approach", *Journal of Public Economics* 87:2611-2637.

[3] Borel, E. (1921), "La théorie du jeu les équations intégrales à noyau symétrique", *Comptes Rendus de l'Académie* 173:1304-1308; English translation by L. Savage (1953), "The theory of play and integral equations with skew symmetric kernels", *Econometrica* 21:97-100.

[4] Borel, E., and J. Ville (1938), *Application de la théorie des probabilitiés aux jeux de hasard*, Gauthier-Villars, Paris; reprinted in Borel E., and A. Chéron, *Théorie mathematique du bridge à la portée de tous*, Editions Jacques Gabay, Paris, 1991.

[5] Feldman, M., and C. Gilles (1985), "An expository note on individual risk without aggregate uncertainty," *Journal of Economic Theory* 35:26-32.

[6] Gross, O., and R. Wagner (1950), "A continuous Colonel Blotto game", RM-408, RAND Corp., Santa Monica.

[7] Harris, J. W., and H. Stocker (1998), "Regular n-gons (Polygons)," in *Handbook of Mathematics and Computational Science*, Springer-Verlag, New York.

[8] Hillman, A. L., and J. G. Riley (1989), "Politically contestable rents and transfers," *Economics and Politics* 1:17-39.

[9] Judd, K. L. (1985), "The law of large numbers with a continuum of iid random variables," *Journal of Economic Theory* 35:19-25.

[10] Laslier, J. F. (2002), "How two-party competition treats minorities," *Review of Economic Design* 7:297-307.

[11] Laslier, J. F., and N. Picard (2002), "Distributive politics and electoral competition", *Journal of Economic Theory* 103:106-130.

[12] Lizzeri, A., and N. Persico (1998), "The provision of public goods under alternative electoral incentives," CARESS Working Paper No. 98-08, University of Pennsylvania, 1998.

[13] Lizzeri, A., and N. Persico (2001), "The provision of public goods under alternative electoral incentives," *American Economic Review* 91:225-239.

[14] Lockwood, B. (2002), "Distributive politics and the costs of centralization", *Review of Economic Studies* 69:313-337.

[15] Musgrave, R. M. (1959), *The Theory of Public Finance*, McGraw-Hill, New York.

[16] Myerson, R. B. (1993), "Incentives to cultivate favored minorities under alternative electoral systems," *American Political Science Review* 87:856-869.

[17] Nelson, R. B. (1999), *An Introduction to Copulas*, Springer, New York.

[18] Oates, W. (1972), *Fiscal Federalism*, Harcourt Brace, New York.

[19] Schweizer, B., and A. Sklar (1983), *Probabilistic Metric Spaces*, North-Holland, New York.

[20] Sklar, A. (1959), "Fonctions de répartition à n dimensions et leurs marges," *Publ. Inst. Statist. Univ. Paris* 8:229-231.

2.5 Appendix

The proof of uniqueness of the equilibrium univariate marginal distributions in Theorem 2, which is contained in the following lemmas, establishes that there exists a one-to-one correspondence between the equilibrium univariate marginal distributions of the utilities that result from any taxes/transfers and the equilibrium distributions of bids from a unique set of two-bidder independent and identical simultaneous all-pay auctions. The uniqueness of the equilibrium univariate marginal distributions then follows from the characterization of the all-pay auction by Hillman and Riley (1989) and Baye, Kovenock and de Vries (1996). In the discussion that follows, $\bar{s}_i^j$ is the upper bound of candidate i's distribution of transfers for district j, and N is the set of all voters.

Lemma 1: In any equilibrium, $\{P_{-i}, P_{-i}\}$, $\bar{s}_i^{z(j)} < G_j$ $\forall$ i and $z(j)$.

Proof: By way of contradiction suppose that there exists an equilibrium in which $\bar{s}_A^{z(j)} > G_j$ for at least one $z(j)$. A feasible strategy for candidate B is to play the equilibrium described in Theorem 2. Letting $\bar{\mathcal{Z}}$ denote the set of $z(j)$ for which $\bar{s}_A^{z(j)} > G_j$, the vote share for candidate A, π_A is

$$\begin{gathered}\pi_A = \tfrac{1}{n}\sum_{i=1}^{k} n_i \tfrac{\alpha_A^j}{2} + \tfrac{1}{2n}\sum_{z(j)\notin\bar{\mathcal{Z}}} \int_0^1 x dF_A^{z(j)} + \\ \tfrac{1}{2n}\sum_{z(j)\in\bar{\mathcal{Z}}} \int_0^{\bar{s}_A^{z(j)}} F_B^{z(j)}(x)\, dF_A^{z(j)} + \\ \tfrac{1-\alpha_A^j}{2n}\sum_{z(j)\in\bar{\mathcal{Z}}} \int_{G_j}^{\bar{s}_A^{z(j)}} F_B^{z(j)}(x - G_j)\, dF_A^{z(j)}\end{gathered}$$

From equation (2) in Proposition 1 it follows that

$$\begin{gathered}\tfrac{1}{n}\sum_{i=1}^{k} n_i \tfrac{\alpha_A^j}{2} + \tfrac{1}{2n}\sum_{z(j)\notin\bar{\mathcal{Z}}} \int_0^1 x dF_A^{z(j)} \leq \\ \tfrac{1}{2} - \tfrac{1}{2n}\sum_{z(j)\in\bar{\mathcal{Z}}} \int_0^{\bar{s}_A^{z(j)}} x dF_A^{z(j)}\end{gathered}$$

Thus after simplifying we have that

$$\begin{gathered}\pi_A \leq \tfrac{1}{2} - \tfrac{1}{2n}\sum_{z(j)\in\bar{\mathcal{Z}}} \int_1^{G_j} (x-1)\, dF_A^{z(j)} + \\ \tfrac{1}{2n}\sum_{z(j)\in\bar{\mathcal{Z}}} \int_{G_j}^{\bar{s}_A^{z(j)}} \left(-\alpha_A^j x - \left(1-\alpha_A^j\right) G_j + 1\right) dF_A^{z(j)} < \tfrac{1}{2}\end{gathered}$$

The argument for player B is symmetric. Q.E.D.

From Proposition 1 we know that the Lagrangian of candidate i's optimization problem is

$$\max_{\left\{L_i^j,\left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k} \frac{1}{n}\sum_{j=1}^k n_j\left(1-\alpha_{-i}^j-\lambda_i\right)\alpha_i^j+$$
$$\frac{1}{n}\sum_{j=1}^k\sum_{z\in N_j}\left[\lambda_i\int_0^\infty\left[\frac{1-\alpha_{-i}^j-\alpha_i^j+2\alpha_{-i}^j\alpha_i^j}{\lambda_i}F_{-i}^{z(j)}(x)-x\right]dF_i^{z(j)}\right]+\lambda_i n$$

where the set of univariate marginal distribution functions

$$\left\{L_i^j,\left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k$$

satisfy the constraint that there exists a $n+k$-copula, C, such that the support of the $n+k$-variate distribution

$$C\left(\left\{L_i^j,\left\{F_i^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k\right)$$

is contained in $\left\{\left\{\iota^j,\left\{t^{z(j)}\right\}_{z\in N_j}\right\}_{j=1}^k \mid \sum_{j=1}^k\left(\iota^j n_j+\sum_{z\in N_j}t^{z(j)}\right)=n\right\}$.

The next three lemmas follow along the lines of the proofs in Baye, Kovenock, and de Vries (1996).

Lemma 2: For each $z(j)\in N$, $\bar{s}_A^{z(j)}=\bar{s}_B^{z(j)}=\bar{s}^{z(j)}$.

Lemma 3: In any equilibrium, $\{P_i,P_{-i}\}$, no $F_i^{z(j)}$ can place an atom in the half open interval $\left(0,\bar{s}^{z(j)}\right]$.

Lemma 4: For each $z(j)\in N$ and for each $i\in\{A,B\}$, $\frac{1-\alpha_A^j-\alpha_B^j+2\alpha_A^j\alpha_B^j}{\lambda_i}F_i^{z(j)}(x)-x$ is constant $\forall\ x\in\left(0,\bar{s}^{z(j)}\right]$.

The following lemma characterizes the relationship between λ_A and λ_B.

Lemma 5: In equilibrium $\lambda_A=\lambda_B\frac{n-\sum_{j=1}^k n_j\alpha_B^j}{n-\sum_{j=1}^k n_j\alpha_A^j}$.

Proof: In any equilibrium each candidate must use their entire budget, thus

$$\begin{aligned}&\sum_{j=1}^k\left(n_j\alpha_A^i+\sum_{z(j)\in N_j}\int_0^{\bar{s}^{z(j)}}xdF_A^{z(j)}\right)\\&=\sum_{j=1}^k\left(n_j\alpha_B^i+\sum_{z(j)\in N_j}\int_0^{\bar{s}^{z(j)}}xdF_B^{z(j)}\right)\\&=n\end{aligned}\qquad(2.4)$$

But, from lemmas 3 and 4, it follows that for all $z(j) \in N$

$$dF_A^{z(j)}(x) = \frac{\lambda_B}{1 - \alpha_A^j - \alpha_B^j + 2\alpha_A^j \alpha_B^j} dx \tag{2.5}$$

for all $x \in \left(0, \bar{s}^{z(j)}\right]$, and

$$dF_B^{z(j)}(x) = \frac{\lambda_A}{1 - \alpha_A^j - \alpha_B^j + 2\alpha_A^j \alpha_B^j} dx \tag{2.6}$$

for all $x \in \left(0, \bar{s}^{z(j)}\right]$. Substituting equations 5 and 6 into equation 4 we have

$$\sum_{j=1}^k n_j \alpha_A^j + \lambda_B \sum_{j=1}^k \frac{1}{1-\alpha_A^j-\alpha_B^j+2\alpha_A^j\alpha_B^j} \sum_{z(j)\in N_j} \int_0^{\bar{s}^{z(j)}} x dx =$$
$$\sum_{j=1}^k n_j \alpha_B^j + \lambda_A \sum_{j=1}^k \frac{1}{1-\alpha_A^j-\alpha_B^j+2\alpha_A^j\alpha_B^j} \sum_{z(j)\in N_j} \int_0^{\bar{s}^{z(j)}} x dx = n.$$

The result follows immediately. Q.E.D.

Since $\bar{s}^{z(j)} = \bar{s}^{-z(j)}$ for all $z(j)$ and $-z(j) \in N_j$, we can define $\bar{s}^j \equiv \bar{s}^{z(j)}$. The following Lemma establishes the value of $\bar{s}^j$.

Lemma 6: $\bar{s}^j = \left(1 - \alpha_A^j - \alpha_B^j + 2\alpha_A^j \alpha_B^j\right) \times \min\left\{\frac{1}{\lambda_A}, \frac{1}{\lambda_B}\right\}$.

Proof: From lemmas 4 and 5, we know that for each candidate i and voter $z(j)$ in district j

$$\frac{1 - \alpha_A^j - \alpha_B^j + 2\alpha_A^j \alpha_B^j}{\lambda_i} F_{-i}^{z(j)}(x) - x$$

is constant $\forall\, x \in \left(0, \bar{s}^{z(j)}\right]$. It then follows that candidate i would never use a strategy that provides transfers in $\left(\frac{1-\alpha_A^j-\alpha_B^j+2\alpha_A^j\alpha_B^j}{\lambda_i}, \infty\right)$ since an offer of zero strictly dominates such a strategy. The result follows directly. Q.E.D.

The following lemma establishes that $\alpha_i^j \equiv \alpha_i\ \forall\ i$ and j.

Lemma 7: In equilibrium, for each candidate i $\alpha_i^j \equiv \alpha_i\ \forall\ j$.

Proof: From the Lagrangian of candidate j's maximization problem the first order condition with respect to α_i^j is

$$n_j\left(1 - \alpha_{-i}^j - \lambda_i\right) + \sum_{z(j)\in N_j} \left(-1 + 2\alpha_{-i}^j\right) \int_0^{\bar{s}^j} F_{-i}^{z(j)}(x)\, dF_i^{z(j)} = 0 \tag{2.7}$$

From Lemmas 3 and 4, and noting that

$$\int_0^{\bar{s}^j} F_{-i}^j(x)\, dx = \bar{s}^j - \frac{\lambda_i \left(\bar{s}^j\right)^2}{2\left(1 - \alpha_A^j - \alpha_B^j + 2\alpha_A^j \alpha_B^j\right)},$$

equation 7 can be written as

$$n_j\left(1-\alpha^j_{-i}-\lambda_i\right) \\ +n_j\left(-1+2\alpha^j_{-i}\right)\left(\bar{s}^j-\frac{\lambda_i\left(\bar{s}^j\right)^2}{2\left(1-\alpha^j_A-\alpha^j_B+2\alpha^j_A\alpha^j_B\right)}\right)\left(\frac{\lambda_{-i}}{1-\alpha^j_A-\alpha^j_B+2\alpha^j_A\alpha^j_B}\right)=0$$

From Lemma 6, we can assume, without loss of generality, that $\bar{s}^j = \frac{1-\alpha^j_A-\alpha^j_B+2\alpha^j_A\alpha^j_B}{\lambda_{-i}}$ $\forall\ j$. Thus equation (7) becomes

$$n_j\left(1-\alpha^j_{-i}-\lambda_i\right)+n_j\left(-1+2\alpha^j_{-i}\right)\left(1-\frac{\lambda_i}{2\lambda_{-i}}\right)=0. \qquad (2.8)$$

The result follows directly from the fact that equation (8) holds for all j. Q.E.D.

Lemma 8: For each candidate i, there exists a unique value for λ_i.

Proof: By symmetry of the parties payoffs and from Lemmas 5 and 7 we have that

$$(1-\alpha_{-i})\,\alpha_i+(1-\alpha_{-i}-\alpha_i+2\alpha_{-i}\alpha_i)\left(1-\frac{1-\alpha_{-i}}{2(1-\alpha_i)}\right)= \\ (1-\alpha_i)\,\alpha_{-i}+(1-\alpha_{-i}-\alpha_i+2\alpha_{-i}\alpha_i)\left(\frac{1-\alpha_{-i}}{2(1-\alpha_i)}\right)$$

It follows directly $\alpha_i=\alpha_{-i}=\frac{1}{2}$.

From equation (8) it follows that $\lambda_i=\frac{1}{2}$. Q.E.D.

This completes the proof of Theorem 2.

3. Electoral Poaching and Party Identification Joint Work with Dan Kovenock

In the model of redistributive politics, political parties compete for representation in a legislature by simultaneously announcing binding commitments as to how they will allocate a budget across voters. Each voter votes for the party offering the highest level of utility, and each party's payoff is its representation in the legislature, which under proportional representation is equal to the fraction of votes received by that party. Originally formulated by Myerson (1993), the model has served as a fundamental tool in the analysis of electoral competition. In recent years, the model has attracted renewed interest through its application to the study of the inequality created by political competition (Laslier (2002), Laslier and Picard (2002)), incentives for generating budget deficits (Lizzeri (2002)), inefficiency of public good provision (Lizzeri and Persico (2001,2002)), and campaign spending regulation (Sahuguet and Persico (2004)).

This paper extends the model of redistributive politics to allow for heterogeneous voter loyalties to political parties and shows that this has important implications for the nature of redistributive competition. Voters are distinguished by the party with which they identify, if any, and the intensity of their attachment, or "loyalty," to that party. We assume that parties are able to perfectly discriminate across voters by their party affiliation and the intensity of their attachment (including the set of "swing voters" who have no attachment to either party). Parties compete by simultaneously announcing offer distributions to each of the identified voter segments. When integrated over all segments, each party's offer distributions must satisfy a common aggregate budget constraint.[1] As in Myerson, each voter is assumed to vote (sincerely) for the party that offers the higher level of utility which, in our model, reflects both the transfer offered and the voter's loyalty.

We completely characterize the unique Nash equilibrium of this model and, explore its qualitative nature. In equilibrium, within any given voter segment, the expected transfers from the two parties' offer distributions are identical. However, we find that voters pay a price for being loyal to a party. For a given distribution of voters' attachments to political parties, the expected transfer that voters receive is strictly decreasing in the voters' intensity of attachment (regardless of party affiliation). This monotonicity of transfers also translates into a monotonicity of utility. Although the expected utility provided by a party's redistribution schedule is identical for all of its loyal voter segments and equal to the expected utility that the swing voters receive from each party's redistribution schedule, the expected utility that a party's loyal voters receive from the opposition party's redistribution schedule is decreasing in the voters' level of attachment.

Moreover, we find that the parties have an incentive to target or "poach" a subset of the opposition party's loyal voters, in an effort to induce those voters to vote against their party. By "poaching" we mean a strategy of targeting each segment of the opposition party's loyal voters with a redistribution schedule that "freezes out" a portion of the segment with a zero transfer, but gives the remaining voters in the segment non-zero transfers which are higher in expectation than the opposition party's offers to the same segment. This captures the notion that a party may try to selectively induce a strict subset of the opposition's loyal voters to defect by offering them a higher transfer.

[1]As in Myerson (1993) each offer distribution is a probability distribution over the nonnegative real numbers with the measure over each interval interpreted as the fraction of the particular loyal voter segment for whom the party's transfer has value in that interval. Since we assume a continuum of voters in each segment and offers that are independent across voters (each voter takes an independent draw from the offer distribution) we may appeal to Judd (1985) in assuming that the aggregate budget constraint holds with probability one and not just in expectation.

To facilitate our analysis we also construct a natural measure of "party strength" based on both the sizes and intensities of a party's loyal voter segments and show how party behavior varies with the two parties' strengths. We demonstrate that each party's vote share is increasing (decreasing) in its own (opponent's) party strength. We also find that as the opposition party's strength increases, a party's equilibrium redistribution schedule freezes out a larger set of the opposition's loyal voters and gives a higher expected transfer to those not frozen out. The party's own loyal voter segments also receive a higher expected transfer. Although it is not obvious from these effects, the level of inequality (as measured by the Gini-coefficient) in a party's equilibrium redistribution schedule is also increasing in the opposition party's strength.

As is common in models of electoral competition, the policy implemented by the legislature is assumed to be a probabilistic compromise of the parties' equilibrium redistribution schedules. The probability that a party's schedule is adopted is proportional to the size of its legislative contingent.[2] From the characterization of equilibrium described above, it immediately follows that for a given distribution of voters' attachments to the political parties, the equilibrium expected transfers and resulting expected utilities from the implemented policy are highest for swing voters and strictly decreasing in the intensity of attachment.[3] Moreover, defining the "level of partisanship" as the sum of the parties' strengths, we find that partisanship preserving transformations of the electorate that increase the strength of party i at the expense of party $-i$ result in party i's loyal voters receiving higher expected utilities and party $-i$'s loyal voters receiving lower expected utilities from the implemented policy.

[2]This interpretation is due to Grossman and Helpman (1996). Probabilistic compromise can also be viewed as a system under which each party distributes a fraction of the budget, proportional to its representation in the legislature, according to its announced schedule. This approach is taken in Myerson (1993).

[3]For expected transfers this result holds regardless of party affiliation; for expected utilities it holds within each party.

We also develop a measure of "political polarization" that is increasing in the sum and symmetry of the parties' strengths and show that the expected ex-post inequality in utilities (as measured by the expected Gini-coefficient) under the implemented policy is increasing in political polarization. In particular, partisanship preserving transformations of the electorate that decrease the difference in the parties' strengths increase the expected ex-post inequality in utilities of the implemented policy. Hence, for a given level of partisanship, the expected ex-post inequality in utilities is maximized when the parties are of equal strength. In addition, holding constant the difference in the parties' strengths, the expected ex-post inequality in utilities increases as the level of partisanship increases. That is, higher levels of partisanship and more symmetry in the parties' strengths generate inequality.

Two related papers are Laslier (2002) and Dixit and Londregan (1996). Laslier (2002) examines the issue of tyranny of the majority[4] in a model of redistributive politics with a segmented homogeneous electorate and intra-segment homogeneity in a party's offers. That is, within each voter segment, a party's offer distribution is assumed to be degenerate with all mass on the fixed offer for that segment (although offers may vary across segments). In this context, Laslier finds that there is no tyranny of the majority as long as there does not exist a segment that contains over half of the voters. However, if any segment contains over half of the voters, each party uses its entire budget on that segment, thereby freezing out the remaining voters.

Our model extends the Laslier model in two ways. First, we allow for a heterogeneous electorate, partitioned into distinct segments of homogeneous voters. Second, we allow for intra-segment heterogeneity in a party's offers, as represented by the (general, non-decreasing) segment specific offer distributions. Since our model as-

[4]Tocqueville describes tyranny of the majority as follows,

> "For what is a majority taken collectively if not an individual with opinions and, more often than not, interests contrary to those of another individual known as the minority. Now, if you are willing to concede that a man to whom omnipotence has been granted can abuse it to the detriment of his adversaries, why will you not concede that the same may be true of a majority?" (pp. 288-289)

sumes that the implemented policy is a probabilistic compromise of the parties' redistribution schedules, a natural analogue of "tyranny of the majority" is the degree to which the implemented policy tyrannizes a minority by driving them down to their reservation utility level. In our model this arises when the implemented policy freezes out voters by giving them an ex-post transfer of zero.[5] Indeed, under our assumption that the probability that a party's schedule is adopted is equal to the size of its vote share, the expected measure of the set of voters receiving a transfer of zero under the implemented policy is proportional to our measure of polarization. That is, *polarization leads to tyranny.*[6]

Dixit and Londregan (1996, Henceforth D-L) is perhaps closest to our paper in focus. Both papers assume voters derive utility from redistribution and party identification. Both assume a heterogeneous electorate, partitioned into distinct voter segments. While D-L assume a non-degenerate distribution of voter attachments within each segment (represented by a segment-specific density), in our model, voters within a given segment are homogeneous, corresponding to perfect discrimination by party affiliation and intensity of attachment. Moreover, like Laslier, D-L assume intra-segment homogeneity in a party's offers. That is, within each voter segment, a party's offer distribution is assumed to be degenerate with all mass on the fixed offer for that segment. This, together with intra-segment heterogeneity of voters, precludes the ability to directly target voters by intensity of attachment. In contrast, our model allows for intra-segment heterogeneity in a party's offers, as represented by

[5]An alternative interpretation of tyranny of the majority refers to an outcome in which a majority receives a higher utility than some designated minority. In the equilibrium in our model, the expected utility, conditional on receiving a positive transfer in the implemented policy, is identical for all voters. Hence, among voters not frozen out there is a form of (conditional) equal treatment. However, within each party, the greater a voter's intensity of attachment, the lower his expected utility from the implemented policy. This arises because parties never freeze out their own loyal voters and the probability that a party's offer distribution freezes out an opposition voter is increasing in that voter's attachment.

[6]This formulation of tyranny would not apply to Myerson's interpretation of probabilistic compromise as a system under which each party distributes a fraction of the budget, proportional to its representation in the legislature, according to its announced schedule. Under this interpretation, no voters would be frozen out ex post in the implemented policy. However, under the implemented policy the unequal treatment (in utilities) of the more loyal voter segments within each party would continue to hold.

a (general, non-decreasing) segment specific offer distribution. Hence, in our model, not only are parties able to directly target voters by party affiliation and intensity of attachment, they are also able to (anonymously) offer discriminate across voters within a given segment.

In section 2 we present the model and characterize the unique Nash equilibrium of the game of redistributive politics with party identification. Section 3 explores the qualitative nature of the equilibrium and presents comparative statics results with respect to changes in measures of party strength, partisanship, and political polarization. Section 4 concludes.

3.1 The Model

Political Parties and the Legislature

Our model extends Myerson's (1993) two-party model of redistributive competition by including heterogeneous voter loyalties to political parties. Two parties, A and B, make simultaneous offers to each of a continuum of voters of unit measure. Each voter votes for the party offering the higher level of utility, and each party's payoff is its representation in the legislature, which under proportional representation is equal to the fraction of votes received by that party. All offers must be nonnegative and each party has a budget of 1, which corresponds to 1 unit of a homogeneous good per voter. Parties are assumed to have complete information regarding the party preferences of all voters. While this is a stylized assumption, this is not an unreasonable benchmark given the high level of organization of modern political parties.[7]

As is commonly assumed in the literature on electoral competition, the legislature implements a policy that is a probabilistic compromise of the parties' redistribution schedules. The policy that the legislature implements is a random variable which

[7]See for example PBS (2004) which discusses the high level of information that national political parties have access to and use to target voters.

takes on party A's equilibrium redistribution schedule with probability equal to party A's equilibrium vote share and takes on party B's equilibrium redistribution schedule with probability equal to party B's equilibrium vote share.

> **Definition:** The *implemented policy* is a random variable that takes on party A's equilibrium redistribution schedule with probability equal to party A's vote share and party B's equilibrium redistribution schedule with probability equal to party B's equilibrium vote share.

Voters

Voters are distinguished by the party with which they identify, if any, and the intensity of their attachment to that party. In this paper, we consider only distributions of voters' attachments to the political parties with support on a finite set of intensities of attachment. Let $\delta_i^j \in (0,1)$ represent the number of units of the homogeneous good that party i must offer a loyal voter in its own loyal segment j in order to make that voter indifferent between the two parties when party $-i$ offers one unit of the homogeneous good.[8] Thus, the utility that each loyal voter in party i's segment j receives from an offer of x^A from party A is

$$u_i^j\left(x^A\right) = \begin{cases} x^A & if\ i = B \\ \frac{x^A}{\delta_A^j} & if\ i = A \end{cases}$$

Define $\alpha_i^j = 1-\delta_i^j$ to be the intensity of attachment of party i's loyal voter segment j. Party A's loyal voters have a finite number, n_A, of different intensities of attachment. Let $\mathcal{A}$ be the set of all indices of intensity of attachment for voters loyal to party A. Each index of intensity $j \in \mathcal{A}$ corresponds to a segment of voters with intensity of attachment α_A^j and measure $m_j > 0$. The *size of party* A is denoted by $M_A = \sum_{j \in \mathcal{A}} m_j$. Similarly, party B's loyal voters have a finite number, n_B, of different intensities of attachment. Let $\mathcal{B}$ be the set of all indices of intensity of attachment

[8]This type of effectiveness advantage originates, to the best of our knowledge, with Lein (1990) and is frequently used in the literature on unfair contests (see for instance: Clark and Riis (2000), Konrad (2002), and Sahuguet and Persico (2004)).

for voters loyal to party B, where $\mathcal{A}$ and $\mathcal{B}$ are disjoint sets. Each index of intensity $k \in \mathcal{B}$ corresponds to a segment of voters with intensity of attachment $\mathfrak{a}_B^k$ and measure $m_k > 0$. The *size of party B* is denoted by $M_B = \sum_{k \in \mathcal{B}} m_k$. There are also swing voters who do not identify with either party. Letting S be the index for no attachment to either party, the utility that each swing voter receives from an offer of x^j from party j is

$$u_S\left(x^j\right) = x^j \quad for\ j = A, B$$

The measure of swing voters is $m_S \equiv 1 - M_A - M_B \geq 0$. To summarize $\left\{\left\{m_j, \mathfrak{a}_A^j\right\}_{j \in \mathcal{A}}, \left\{m_k, \mathfrak{a}_B^k\right\}_{k \in \mathcal{B}}\right\}$ is a feasible distribution of voters' attachments to the political parties if n_A and n_B are finite, $M_A + M_B \leq 1$, and $m_j > 0$ for all $j \in \mathcal{A} \cup \mathcal{B}$.

Each voter votes for the party that provides them the higher utility. Thus each swing voter votes for the party that makes them the higher transfer, while each loyal voter requires a proportionally higher transfer from the rival party in order to induce him to cross over. Representation in the legislature is allocated proportionally. Thus, we normalize each party's representation in the legislature to be equal to the fraction of the votes received by that party.

One simple yet important summary statistic of a party's distribution of loyal voters is the sum across segments of each segment's intensity of attachment weighted by the measure of the set of voters in that segment.

> **Definition:** The *strength of party A* is denoted by $\sigma_A \equiv \sum_{j \in \mathcal{A}} m_j \mathfrak{a}_A^j$. The *strength of party B* is denoted by $\sigma_B \equiv \sum_{k \in \mathcal{B}} m_k \mathfrak{a}_B^k$

Several properties of this summary statistic should be noted. First, holding constant the size of each of a party's loyal segments, the party's strength is strictly increasing in the intensity of the attachment of any of these segments. Second, holding constant the intensity of attachment of each of its loyal segments, the party's strength is strictly increasing in the size of each of these segments. Finally, holding constant a party's size, the party's strength is strictly increasing as loyal voters shift from weaker intensities of attachment to stronger intensities of attachment.

Given the parties' strengths, σ_A and σ_B, it is useful to derive two simple measures for the distribution of voters' attachments to the political parties.

Definition: The *level of partisanship* is the sum of the parties' strengths, which is denoted by $\sigma \equiv \sigma_A + \sigma_B$.

Definition: The *effective strength of party i* is denoted by $\hat{\sigma}_i \equiv \sigma_i - \sigma_{-i}$.

The level of partisanship is the sum across the entire electorate of each segment's intensity of attachment, to either party, weighted by the measure of the set of voters in that segment. The properties of the level of partisanship are similar to those of the parties' strengths. Holding constant the size of each loyal segment, the level of partisanship is strictly increasing in the intensity of attachment of each segment. In addition, the level of partisanship is strictly increasing as loyal voters shift from weaker intensities of attachment to stronger intensities of attachment or as swing voters become affiliated with a political party. The effective strength of party i measures the asymmetry between party i and party $-i$. If the parties have symmetric strengths then each party has an effective strength of 0.

Redistributive Competition

A strategy, which we label a *redistributive schedule* (or *offer distribution*), for party i is a set of cumulative distribution functions,[9] $\{F_i^j\}_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}}$, one distribution function for each segment $j \in \mathcal{A}$ of voters loyal to party A, the segment of swing voters S, and each segment $k \in \mathcal{B}$ of voters loyal to party B. As in Myerson (1993) each $F_i^j(x)$ denotes the fraction of voters in segment j whom party i will offer a transfer less than or equal to x. The only restrictions that are placed on the set of feasible strategies is that each offer must be nonnegative and the set of cumulative distribution functions must satisfy the budget constraint:

$$\sum_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}} m_j \int_0^{\infty} x dF_i^j \leq 1 \tag{3.1}$$

[9]In this case the focus is on the distributions within each segment (marginal distributions) rather than an n-variate joint distribution. As discussed in the appendix, an n-variate joint distribution is trivial to obtain and adds nothing to the problem analyzed here.

Redistributive competition is the one-shot game, which we label

$$G\left(\left\{\{m_j, \alpha_A^j\}_{j\in\mathcal{A}}, \{m_k, \alpha_B^k\}_{k\in\mathcal{B}}\right\}\right),$$

in which parties compete for representation in the legislature by simultaneously announcing redistributive schedules, subject to a budget constraint.

Optimal Strategies

The following theorem characterizes the equilibrium of the redistributive competition game.

Theorem 1: The unique Nash equilibrium of the redistributive competition game $G\left(\left\{\{m_j, \alpha_A^j\}_{j\in\mathcal{A}}, \{m_k, \alpha_B^k\}_{k\in\mathcal{B}}\right\}\right)$ is for each party i to choose offers according to the following distributions. For party A

$$\begin{array}{lll} \forall\ j \in \mathcal{A} & F_A^j(x) = \frac{x}{z\left(1-\alpha_A^j\right)} & x \in \left[0, z\left(1-\alpha_A^j\right)\right] \\ & F_A^S(x) = \frac{x}{z} & x \in [0, z] \\ \forall\ k \in \mathcal{B} & F_A^k(x) = \alpha_B^k + \left(1-\alpha_B^k\right)\frac{x}{z} & x \in [0, z]\,. \end{array}$$

Similarly for party B

$$\begin{array}{lll} \forall\ k \in \mathcal{B} & F_B^k(x) = \frac{x}{z\left(1-\alpha_B^k\right)} & x \in \left[0, z\left(1-\alpha_B^k\right)\right] \\ & F_B^S(x) = \frac{x}{z} & x \in [0, z] \\ \forall\ j \in \mathcal{A} & F_B^j(x) = \alpha_A^j + \left(1-\alpha_A^j\right)\frac{x}{z} & x \in [0, z]\,, \end{array}$$

where $z = \frac{2}{1-\sigma} = \frac{2}{1-\sigma_A-\sigma_B}$. In equilibrium, party A's share of the vote is $\frac{1+\hat{\sigma}_A}{2} = \frac{1+\sigma_A-\sigma_B}{2}$ and party B's share of the vote is $\frac{1+\hat{\sigma}_B}{2} = \frac{1+\sigma_B-\sigma_A}{2}$.

Proof: We begin by showing that this is an equilibrium. First, this is a feasible strategy since:

$$\sum_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}} m_j \int_0^\infty x dF_i^j = 1$$

Then given that party B is following the equilibrium strategy, the vote share $\pi_A(\cdot)$ for party A, when it chooses to provide transfers according to an arbitrary strategy $\{\bar{F}_A^j\}_{j\in\mathcal{A}\cup S\cup\mathcal{B}}$ is:

$$\pi_A\left(\{\bar{F}_A^j, F_B^j\}_{j\in\mathcal{A}\cup S\cup\mathcal{B}}\right) = \sum_{j\in\mathcal{A}} m_j \int_0^\infty F_B^j\left(\tfrac{x}{1-\mathfrak{a}_A^j}\right) d\bar{F}_A^j(x)$$
$$+m_S \int_0^\infty F_B^S(x)\, d\bar{F}_A^S(x)$$
$$+\sum_{k\in\mathcal{B}} m_k \int_0^\infty F_B^k\left(x\delta_B^k\right) d\bar{F}_A^k(x)$$

Since it is never a best response for party A to provide offers outside the support of party B's offers, we have:

$$\pi_A\left(\{\bar{F}_A^j, F_B^j\}_{j\in\mathcal{A}\cup S\cup\mathcal{B}}\right) = \tfrac{1}{z}\sum_{j\in\mathcal{A}} m_j \int_0^{z\left(1-\mathfrak{a}_A^j\right)} x d\bar{F}_A^j(x)$$
$$+\sum_{j\in\mathcal{A}} m_j\mathfrak{a}_A^j + \tfrac{m_S}{z}\int_0^z x d\bar{F}_A^S(x)$$

$$\pi_A\left(\{\bar{F}_A^j, F_B^j\}_{j\in\mathcal{A}\cup S\cup\mathcal{B}}\right) = \tfrac{1}{z}\sum_{j\in\mathcal{A}} m_j \int_0^{z\left(1-\mathfrak{a}_A^j\right)} x d\bar{F}_A^j(x)$$
$$+\sum_{j\in\mathcal{A}} m_j\mathfrak{a}_A^j + \tfrac{m_S}{z}\int_0^z x d\bar{F}_A^S(x)$$

But from the budget constraint given in equation (1) it follows that

$$\pi_A\left(\{\bar{F}_A^j, F_B^j\}_{j\in\mathcal{A}\cup S\cup\mathcal{B}}\right) \leq \frac{1}{z} + \sum_{j\in\mathcal{A}} m_j\mathfrak{a}_A^j = \frac{1+\sigma_A-\sigma_B}{2}$$

which holds with equality if $\{\bar{F}_A^j\}_{j\in\mathcal{A}\cup S\cup\mathcal{B}}$ is the equilibrium strategy. Thus party A's vote share cannot be increased by deviating to another strategy. The argument for party B is symmetric.

In the appendix, the strategic equivalence between two-party games of redistributive politics with segmented voters and independent simultaneous two-bidder all-pay auctions is established. The proof of uniqueness then follows from the arguments appearing in Baye, Kovenock and de Vries (1996). Q.E.D.

The following example illustrates the key features of Theorem 1.

Example: Assume that there are only two types of voters: voters loyal to party A and voters loyal to party B. Let $m_A = \frac{1}{3}$, $\mathfrak{a}_A = \frac{1}{2}$, $m_B = \frac{2}{3}$, and $\mathfrak{a}_B = \frac{3}{4}$. Party A's and party B's strengths are $\sigma_A = \frac{1}{3}\left(\frac{1}{2}\right) = \frac{1}{6}$ and $\sigma_B = \frac{2}{3}\left(\frac{3}{4}\right) = \frac{1}{2}$, respectively. Party A's and party B's equilibrium vote shares are $\frac{1+\hat{\sigma}_A}{2} = \frac{1+\sigma_A-\sigma_B}{2} = \frac{1}{3}$ and $\frac{1+\hat{\sigma}_B}{2} = \frac{1+\sigma_B-\sigma_A}{2} = \frac{2}{3}$, respectively. The transfers and resulting utilities from the unique equilibrium redistribution schedules given by Theorem 1 are shown in Figure 1 below. As party $-i$'s loyal voters' intensity of attachment, $\mathfrak{a}_{-i}$, increases, party i freezes out a larger proportion of $-i$'s loyal voters with a zero transfer. This is represented graphically in Figure 1(a) and 1(b) as shift up of $F_i^{-i}(0) = \mathfrak{a}_{-i}$.

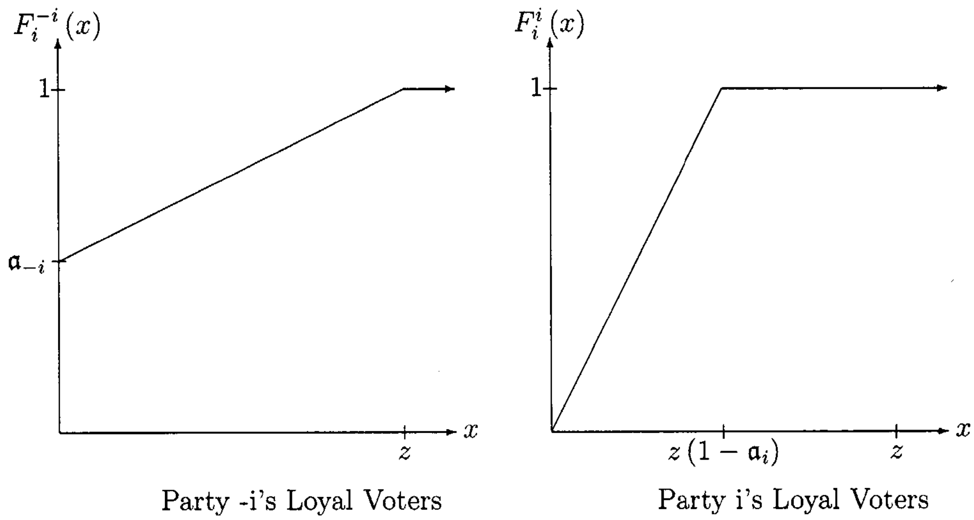

Figure 1(a): Transfers from Party i's Equilibrium Redistribution Schedule

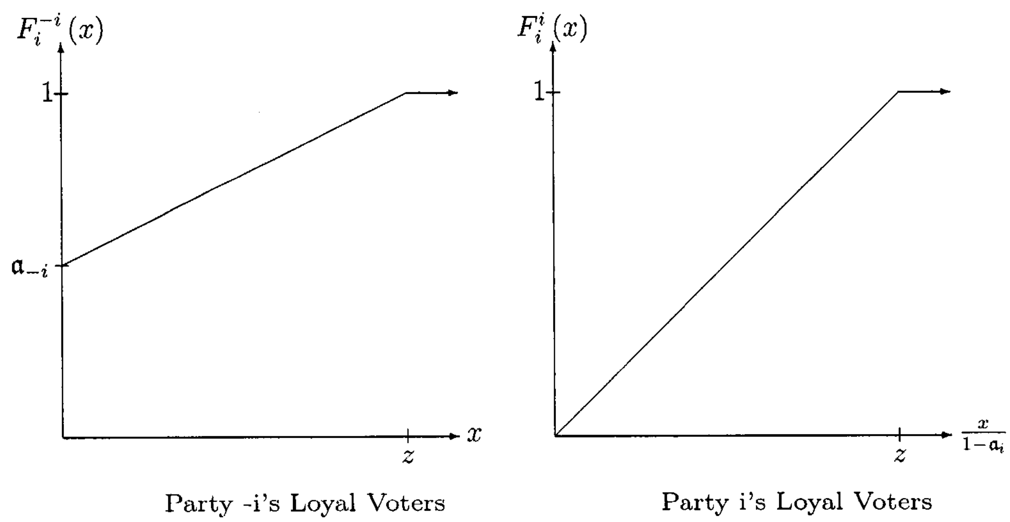

Figure 1(b): Utilities from Party i's Equilibrium Redistribution Schedule

Note that, each party's equilibrium vote share is increasing (decreasing) in its own (opponent's) party strength. Party identification also creates an incentive for parties to utilize a poaching strategy which freezes out a portion of the opposition's loyal voters with a zero transfer, but gives the remaining opposition voters non-zero transfers which are higher in expectation than the opposition party's offers. A similar poaching effect has been addressed in the industrial organization literature on brand loyalty and brand switching. For example, Fudenberg and Tirole (2000)[10] examine a duopoly model of brand loyalty and brand switching where firms try to poach the competitor's loyal consumers. The electoral poaching examined here differs from Fudenberg and Tirole (2000) in that the focus is on redistribution rather than short-term versus long-term contracts.

3.2 Transformations of the Electorate

We now apply Theorem 1 to explore the qualitative nature of the equilibrium and present comparative statics results with respect to changes in measures of party strength, partisanship, and political polarization. We begin with the nature of the equilibrium for a given distribution of voter attachments. In redistributive competition with heterogeneous voter loyalties, each party announces a distribution of offers for each segment of the electorate. In the discussion that follows we refer to the expectation of a party's equilibrium distribution of offers over a segment as that segment's equilibrium expected transfer from the party's redistribution schedule. A segment's equilibrium expected utility from the party's redistribution schedule is similarly defined, as are both the equilibrium transfer and utility from the implemented policy.

Despite the fact that from Theorem 1 the parties' equilibrium redistributive schedules differ in all segments of loyal voters, for each segment, the expected transfer from each party, and thus from the implemented policy, is the same. Furthermore,

[10]See also Lee (1997).

for a given distribution of voters' attachments to the political parties, the expected transfers are highest for the swing segment and are strictly decreasing in the intensity of attachment to a party. Thus, voters with the highest intensity of attachment receive the lowest expected transfers and swing voters receive the highest expected transfers. However, the poaching strategies utilized by the parties freeze out a portion of the opposition party's loyal voters with a zero transfer, and offer the remaining portion of the opposition party's loyal voters non-zero transfers which are higher in expectation than the opposition party's transfers. For each segment, conditional on receiving a positive transfer from the opposition party the expected transfer from the opposition party is equal to the expected transfer of the swing segment.

Corollary 1: Within any given voter segment, the expected transfers from the two parties are identical. For a given distribution of voters' attachments to the political parties, the expected transfers are strictly decreasing in the intensity of attachment (regardless of party affiliation). Conditional on receiving a positive transfer from the opposition party, within each loyal voter segment the expected transfer from the opposition party is equal to that of the swing voter segment.

Proof: From Theorem 1 the swing voters equilibrium expected transfer from each party and from the implemented policy $E^S(\cdot)$ is

$$E^S\left(\left\{\left\{m_j, \alpha_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \alpha_B^k\right\}_{k\in\mathcal{B}}\right\}\right) = \frac{1}{1-\sigma}$$

Similarly, for each segment $j \in \mathcal{A}$ of party A's loyal voters the equilibrium expected transfer from each party and from the implemented policy $E^j(\cdot)$ is

$$E^j\left(\left\{\left\{m_j, \alpha_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \alpha_B^k\right\}_{k\in\mathcal{B}}\right\}\right) = \frac{1-\alpha_A^j}{1-\sigma}$$

Conditional on receiving a positive transfer from party B, for each segment $j \in \mathcal{A}$ of party A's loyal voters the equilibrium expected transfer from party B $E_+^j(\cdot)$ is

$$E_+^j\left(\left\{\left\{m_j, \alpha_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \alpha_B^k\right\}_{k\in\mathcal{B}}\right\}\right) = \frac{1}{1-\sigma}$$

The argument for voters loyal to party B is symmetric. Q.E.D.

The swing voter segment is the most contested segment since neither party has an advantage, and, thus, the equilibrium transfers are the highest in this segment. The presence of voter loyalties to the political parties creates an incentive for the parties to target or poach a subset of the opposition party's loyal voters. However, as a segment's intensity of attachment increases it becomes more difficult for the opposition party to induce a voter in that segment to vote against their party. Thus, the proportion of a segment's loyal voters that the opposition party targets with non-zero transfers is decreasing in the intensity of attachment. As the more attached segments are targeted less by the opposition, the affiliated party optimally diverts resources away from its most attached loyal voter segments to the other segments. This result is independent of the measures of the segments and the parties' strengths.

One difference between our results on expected transfers and the analysis of the resulting utilities is that for loyal voters the expected utilities from the affiliated party's redistribution schedule are higher than the unconditional expected utilities from the opposition party's redistribution schedule. In fact, the expected utility that each segment of loyal voters receives from the affiliated party's redistribution schedule is equal to the expected utility that the swing voters receive from either party's redistribution schedule. In addition, conditional on receiving a positive transfer, the expected utility that each subset of loyal voters receives from the opposition party's redistribution schedule is also equal to the expected utility that the swing voters receive. Thus, since the proportion of a segment's loyal voters that is targeted with non-zero transfers is decreasing in the intensity of attachment, the unconditional expected utility that each segment of loyal voters' receives from the opposition party's redistribution schedule is strictly decreasing in the intensity of attachment.

Corollary 2: For all loyal voter segments, the expected utility from the affiliated party's redistribution schedule and the expected utility conditional on receiving a positive transfer from the opposition party's redistribution schedule are identical and equal to the expected utility that the swing voters receive from either party's redistribution schedule. For a given distribution of voters' attachments to the political parties, loyal voters' unconditional expected utilities from the opposition party's redistribution schedule are strictly decreasing in the intensity of attachment.

Proof: We present the argument for party A's loyal segments. The argument for party B's segments is symmetric. From Theorem 1, for each segment $j \in \mathcal{A}$ of party A's loyal voters the equilibrium expected utility from party A, and the expected utility conditional on receiving a positive transfer from party B, $EU_A^j(\cdot)$ and $EU_{B+}^j(\cdot)$ respectively, are

$$EU_A^j\left(\left\{\left\{m_j, \alpha_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \alpha_B^k\right\}_{k\in\mathcal{B}}\right\}\right) =$$
$$EU_{B+}^j\left(\left\{\left\{m_j, \alpha_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \alpha_B^k\right\}_{k\in\mathcal{B}}\right\}\right) = \frac{1}{1-\sigma}$$

From Corollary 1, this is equal to the expected utility for swing voters, $EU^S = E^S$.

The second part of the corollary follows from the fact that for each segment $j \in \mathcal{A}$ of party A's loyal voters the equilibrium unconditional expected utility from the opposition party's redistribution schedule, $EU_B^j(\cdot)$, is

$$EU_B^j\left(\left\{\left\{m_j, \alpha_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \alpha_B^k\right\}_{k\in\mathcal{B}}\right\}\right) = E^j = \frac{1-\alpha_A^j}{1-\sigma}$$

Q.E.D.

In fact, the equivalence between loyal voters' utilities from the affiliated party's redistribution schedule, the targeted loyal voters' utilities from the opposition party's redistribution schedule, and the swing voters utilities from both schedules is stronger than stated. The distribution of loyal voters' utilities from the affiliated party, the distribution of targeted loyal voters' utilities from the opposition party, and the distributions of swing voters utilities from both parties are identical.

Given these static properties of the equilibrium transfers and resulting utilities we now examine comparative statics with respect to transformations of the electorate. We will focus mainly on two simple transformations of the electorate. The first, *a partisanship preserving transformation of the electorate*, reflects a change in the symmetry of the parties' strengths while holding the level of partisanship constant. The second, *an effective party-strength preserving transformation of the electorate*, reflects a change in the level of partisanship while holding the absolute difference in the parties' strengths constant. These two types of transformations are represented graphically in Figure 2. In (σ_A, σ_B) space, for a given level of partisanship, the set of partisanship preserving transformations forms a line with slope of -1, and for fixed effective party strengths, the set of effective party-strength preserving transformations forms a line with slope of $+1$.

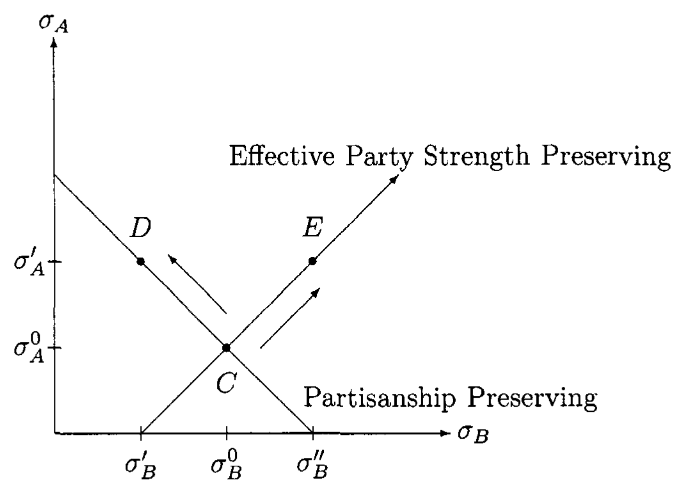

Figure 2: A transformation of the electorate that changes party strengths from $C = (\sigma^0_B, \sigma^0_A)$ to $D = (\sigma'_B, \sigma'_A)$ is a partisanship preserving transformation. A change from C to $E = (\sigma''_B, \sigma'_A)$ is an effective party-strength preserving transformation.

Another transformation that we examine is one that holds constant or fixed the intensities of the attachment to parties, while shifting the electorate across the given set of intensities. The following corollary examines how, given fixed intensities of attachment, partisanship preserving and partisanship increasing transformations of the electorate change each voter segment's expected transfers and utilities. Proof of the corollary follows directly from Corollaries 1 and 2.

Corollary 3: Given $\left\{\{m_j, \alpha_A^j\}_{j\in\mathcal{A}}, \{m_k, \alpha_B^k\}_{k\in\mathcal{B}}\right\}$, a partisanship preserving (resp., increasing) transformation of the electorate that leaves the intensities of attachment, $\{\alpha_A^j\}_{j\in\mathcal{A}}$, $\{\alpha_B^k\}_{k\in\mathcal{B}}$, fixed leaves invariant (resp., increases) the expected transfer and utility received from each party's redistribution schedule by voters within a given voter segment $j \in \mathcal{A}$, $k \in \mathcal{B}$, or S.

Given fixed intensities of attachment, partisanship preserving transformations of the electorate hold constant both the measure of the set of voters that receives a zero transfer from one of the two parties and the expected utility, conditional upon receiving a positive offer, received from either party's redistribution schedule. In addition note that in party i's equilibrium redistribution schedule the proportion of party $-i$'s loyal voter segment j that receives a zero transfer is equal to segment j's intensity of attachment, α_{-i}^j. Thus, the measure of party $-i$'s loyal voters who receive a zero transfer from party i is equal to party $-i$'s strength, σ_{-i}. Regardless of which party gains and which party loses, in a partisanship preserving transformation of the electorate the measure of the set of voters that receive a transfer of 0 from one of the two parties remains invariant. Similarly, partisanship increasing transformations of the electorate result in an increase in the measure of the set of voters who receive a transfer of 0 from one of the two parties.

These corollaries highlight several features of the nature of equilibrium poaching. These are summarized in the following corollary.

Corollary 4: The proportion of loyal voter segment j of party i that receives a transfer of 0 from the redistribution schedule of party $-i$ is α_i^j. Conditional upon receiving a positive transfer from party $-i$, the expected transfer and utility received by a loyal voter in segment j of party i is $\frac{1}{1-\sigma}$. The unconditional expected transfer to a voter in segment j of party i from the redistribution schedule of party $-i$ is $\frac{1-\alpha_i^j}{1-\sigma}$. The proportion of party i's loyal voters who receive a transfer of 0 from party $-i$ is $\frac{\sigma_i}{M_i}$.

This characterization of each party's equilibrium poaching raises the question of how changes in voter loyalty to political parties affect the inequality arising from the equilibrium redistribution schedules. Remarkably, the comparative statics analysis of changes in inequality in the distribution of transfers is considerably more complex than the analysis of changes in the distribution of utilities. The Lorenz curves for the distributions of transfers arising from each party's redistribution schedule are piece-wise quadratic functions that depend critically on each parameter in the distribution of voters' attachments to the parties. The kinks in these curves make it difficult to obtain unambiguous comparative statics results. It turns out that comparative statics results on the inequality in utility are more straightforward. Corollary 5 addresses inequality in the distribution of utilities arising from each party's equilibrium offer distribution as measured by the Gini-coefficient of inequality.

Corollary 5: For each party $i = A, B$, the inequality (as measured by the Gini-coefficient of inequality) arising from the party's equilibrium redistribution schedule is increasing in the opposition party's strength. More precisely, the Gini-coefficient of party i's equilibrium redistribution schedule is $C_i = \frac{1}{3} + \frac{2\sigma_{-i}}{3}$, $i = A, B$.

Proof: From Theorem 1, the measure of the set of voters who receive a utility level from party A's equilibrium redistribution schedule that is less than or equal to x is

$$\tilde{F}_A(x) = \sum_{k \in \mathcal{B}} m_k \mathfrak{a}_B^k + \frac{x}{z}\left(\sum_{k \in \mathcal{B}} m_k \left(1 - \mathfrak{a}_B^k\right) + \sum_{j \in \mathcal{A} \cup \mathcal{S}} m_j\right)$$

for $x \in [0, z]$. Simplifying, $\tilde{F}_A(x) = \sigma_B + \frac{x}{z}(1 - \sigma_B)$ for $x \in [0, z]$. By definition the Lorenz curve for $\tilde{F}_A$ is

$$L_A(y) = \frac{\int_0^y \tilde{F}_A^{-1}(x)dx}{\int_0^1 \tilde{F}_A^{-1}(x)dx}, \quad y \in [0, 1],$$

which is equivalent to

$$L_A(y) = \begin{cases} 0 & if\ y \in [0, \sigma_B] \\ \frac{(y-\sigma_B)^2}{(1-\sigma_B)^2} & if\ y \in (\sigma_B, 1] \end{cases}$$

By definition, the Gini-coefficient for $\tilde{F}_A$ is

$$C_A\left(\left\{\left\{m_j, \delta_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \delta_B^k\right\}_{k\in\mathcal{B}}\right\}\right) = 1 - 2\int_{\sigma_B}^{1} L_A(x)\,dx.$$

Simplifying we have $C_A = \frac{1}{3} + \frac{2\sigma_B}{3}$. It follows that $\frac{\partial C_A}{\partial \sigma_B} > 0$. A similar argument establishes the property for party B's equilibrium redistribution schedule. Q.E.D.

Party i has an incentive to target a different proportion of the voters from each of party $-i$'s loyal segments. As the intensity of attachment of a given segment of $-i$'s voters increases, the proportion of that segment that receives a transfer of 0 increases. As a result, the aggregate inequality in party i's equilibrium redistribution schedule increases.

More generally, as Corollary 5 states, any change in the distribution of voters' attachments to the political parties that leads to an increase in the strength of party $-i$, results in an increase in the aggregate inequality of party i's equilibrium redistribution schedule. Moreover, freezing out by party i increases in the sense that the measure of party $-i$'s loyal voters that receive a transfer of 0 from party i increases.

Given the assumption that the legislature implements a probabilistic compromise of the parties' equilibrium redistribution schedules, we can also examine the expected utilities and the expected ex-post inequality of utilities from the implemented policy. To measure changes in the expected utility from the implemented policy, we must take into account changes both in the level of partisanship and in the parties' effective strengths. In particular, for fixed intensities of attachment, partisanship preserving transformations of the electorate that increase the strength of party i increase party i's loyal voters' expected utilities from the implemented policy and decrease party $-i$'s loyal voters' expected utilities from the implemented policy. Conversely, effective party-strength preserving transformations of the electorate that increase the level of partisanship increase all voters' expected utilities.

Corollary 6: Given fixed intensities of attachment to the parties, partisanship preserving transformations of the electorate that increase the strength of party i increase party i's loyal voters' expected utilities and decrease party $-i$'s loyal voters' expected utilities from the implemented policy. In addition, effective party-strength preserving transformations of the electorate that increase the level of partisanship increase all voters' expected utilities from the implemented policy.

Proof: From Theorem 1, for each segment $j \in \mathcal{A}$ of party A's loyal voters the equilibrium expected transfer from the policy implemented by the legislature $EU^j(\cdot)$ is

$$EU^j\left(\left\{\left\{m_j, \mathfrak{a}_A^j\right\}_{j\in\mathcal{A}}, \left\{m_k, \mathfrak{a}_B^k\right\}_{k\in\mathcal{B}}\right\}\right) = \left(\frac{1+\hat{\sigma}_A}{2}\right)\left(\frac{1}{1-\sigma}\right) + \left(\frac{1-\hat{\sigma}_A}{2}\right)\left(\frac{1-\mathfrak{a}_A^j}{1-\sigma}\right)$$

which is increasing in $\hat{\sigma}_A$ and thus decreasing in $\hat{\sigma}_B = -\hat{\sigma}_A$. The argument for voters loyal to party B is symmetric.

The second part of the corollary follows from the fact that for each segment $j \in \mathcal{A}$

$$\frac{\partial EU^j}{\partial \sigma} > 0$$

The argument for swing voters and voters loyal to party B follows directly. Q.E.D.

The implications of these results for the expected ex-post inequality of utilities from the implemented policy are examined in the following corollary. We use the expected Gini-coefficient to measure expected ex-post inequality and refer to the expected Gini-coefficient as the "*aggregate inequality.*"

Corollary 7: Partisanship preserving transformations of the electorate that increase the symmetry in the parties' strengths increase the aggregate inequality of the implemented policy. Moreover, for a given level of partisanship, σ, the aggregate inequality arising from the implemented policy is maximized when the parties are of equal strength, $\sigma_A = \sigma_B$. Conversely, effective party-strength preserving transformations of the electorate that increase the level of partisanship increase the aggregate inequality of the implemented policy.

Proof: From Corollary 3, the aggregate inequality arising from the implemented policy is

$$I(\sigma_A, \sigma_B) = \left(\frac{1+\hat{\sigma}_A}{2}\right)\left(\frac{1+2\sigma_B}{3}\right) + \left(\frac{1-\hat{\sigma}_A}{2}\right)\left(\frac{1+2\sigma_A}{3}\right)$$

Simplifying we have $I(\sigma_A, \sigma_B) = \frac{1}{3} + \frac{\sigma-(\hat{\sigma}_A)^2}{3}$.

The first and third parts of the corollary follow directly. The second part follows from the fact that for a given level of partisanship, σ, $I(\sigma_A, \sigma_B)$ is maximized when $\hat{\sigma}_A = 0$, or $\sigma_A = \sigma_B = \frac{\sigma}{2}$. Q.E.D.

Hence, for a given level of partisanship symmetry in party strength generates inequality. Similarly, for given effective party-strengths, partisanship generates inequality.

Our results on party strength and inequality are closely related to issues arising in the literature on polarization.[11] Although much of this literature deals with the distribution of income, its tenets can be adapted to our context of redistributive politics. An interesting question that arises is whether there is a simple measure, of "political polarization," defined over the primitives of the model, with the property that the aggregate inequality from the implemented policy is increasing in the measure. It turns out that the answer is yes. Indeed, we base this measure solely on the party strengths. Setting

$$P(\sigma_A, \sigma_B) \equiv \sigma - (\hat{\sigma}_A)^2 = \sigma - (\hat{\sigma}_B)^2$$

it is easily demonstrated that the aggregate inequality in utilities arising from the implemented policy is increasing in $P(\cdot, \cdot)$.

Corollary 8: The aggregate inequality in utilities arising from the implemented policy is increasing in the measure of political polarization $P(\sigma_A, \sigma_B)$.

[11] See for example: Esteban and Ray (1994), Wolfson (1994), Wang and Tsui (2000), and Rodriguez and Salas (2003).

The level curves of the political polarization measure and the aggregate inequality of utilities from the implemented policy are shown in Figure 3 below.

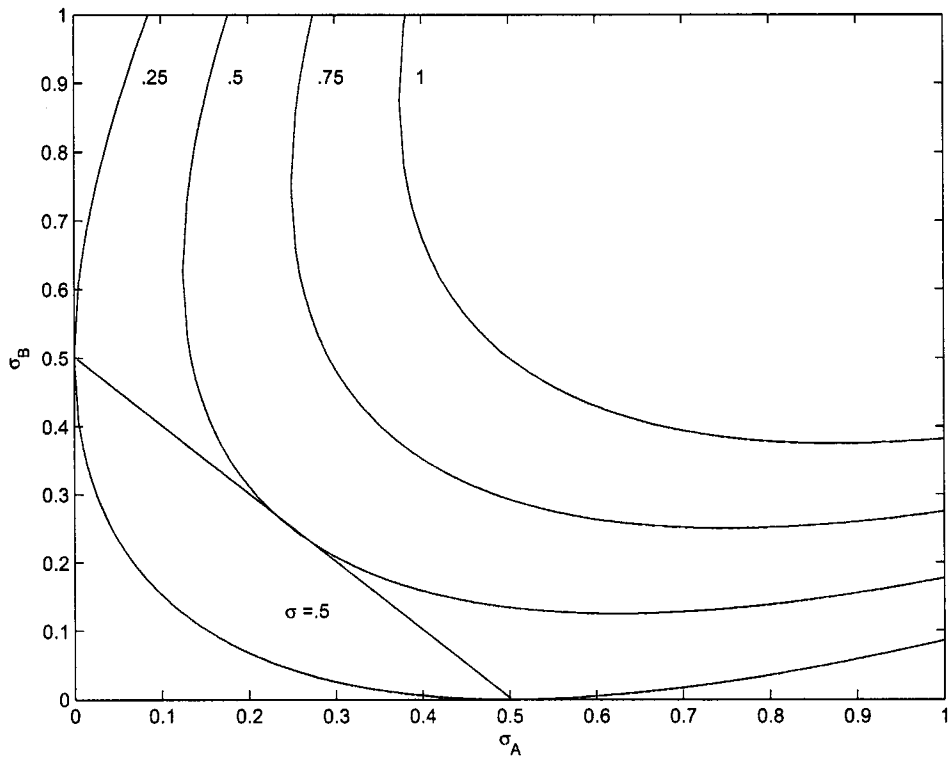

Figure 3: Level Curves of Political Polarization

Several properties of these level curves should be mentioned. First a level of partisanship defines a 'budget' line over possible combinations of party strengths. Thus, the properties of aggregate inequality from the implemented policy addressed in Corollary 7 can be seen graphically in Figure 3. Second, given that the parties have symmetric strengths, an increase in either party's strength increases polarization and thus aggregate inequality. That is

$$\frac{\partial I}{\partial \sigma_i}|_{\hat{\sigma}_A=0} > 0.$$

Furthermore, for $\sigma_i < \frac{1}{2} + \sigma_{-i}$, a small increase in party i's strength increases the aggregate inequality arising from the implemented policy. That is

$$\frac{\partial I}{\partial \sigma_i}|_{\sigma_i<\frac{1}{2}+\sigma_{-i}} > 0.$$

Our results on inequality and political polarization are closely related to the incentive, created by party identification, to freeze out a portion of the opposition party's loyal voters with a zero transfer. Freezing out opposition voters is also closely related to the classic issue of the tyranny of the majority, in which a majority of voters expropriates from a minority. In Laslier (2002), a minority is frozen out only if a single voter segment contains a majority of voters. In contrast, in our model the expected measure of the set of voters frozen out by the implemented policy depends on the parties' strengths and is increasing in the level of political polarization. That is tyranny is increasing in polarization.

Corollary 9: The expected measure of the set of voters that receive a zero transfer from the implemented policy is $\frac{P(\cdot,\cdot)}{2}$. That is tyranny is increasing in the political polarization measure $P(\cdot,\cdot)$.

Proof: From Theorem 1, for each segment $j \in \mathcal{A}$ of party A's loyal voters the probability of receiving a zero transfer from the implemented policy is $\frac{1-\hat{\sigma}_A}{2}\left(\alpha_A^j\right)$. Similarly for each segment $k \in \mathcal{B}$ of party B's loyal voters the probability of receiving a zero transfer from the implemented policy is $\frac{1+\hat{\sigma}_A}{2}\left(\alpha_B^k\right)$. The result follows directly. Q.E.D.

3.3 Conclusion

This paper extends Myerson's (1993) model of redistributive politics to allow for heterogeneous voter loyalties to political parties. Parties segment voters by the party with which they identify, if any, and the intensity of their attachment, or "loyalty," to that party. We find that voters pay a price for party loyalty. For a given distribution of voters' attachments to the political parties, in the implemented policy, the segment of swing voters has the highest expected transfer and expected utility, and the expected transfers and utilities for loyal voter segments are strictly decreasing in the intensity of attachment. Using our measure of "party strength," based on both the sizes and intensities of attachment of a party's loyal voter segments, we demonstrate that each party's representation in the legislature is increasing (decreasing) in its own (opponent's) party strength. In addition, parties poach a subset of the opposition party's loyal voters, in an effort to induce those voters to vote against the opposition party. The level of inequality in and the size of the set of opposition party voters frozen out by a party's equilibrium redistribution schedule are increasing in the opposition party's strength.

We also develop a measure of "political polarization" that is increasing in the sum and symmetry of the party strengths, and find that aggregate inequality is increasing in political polarization. That is, higher levels of partisanship and more symmetry in the parties' strengths generate inequality. In addition, the expected measure of the set of voters that receive a zero transfer (and, hence, their secure utility level) from the implemented policy is increasing in the level of political polarization. In this sense polarization increases tyranny.

There are several potential directions for future research based on our model that appear to be particularly fruitful. The model can be applied to shed light on topics previously studied in the redistributive politics literature, such as candidate valence issues. In addition, this paper's focus on identifiable voter segments is immediately applicable to the study of transfers targeted by geographical region or other identifiable characteristics.

3.4 Bibliography

[1] Baye, M. R., D. Kovenock, and C. G. de Vries (1996), "The all-pay auction with complete information," *Economic Theory* 8:291-305.

[2] Clark, D. J., C. Riis (2000), "Allocation efficiency in a competitive bribery game," *Journal of Economic Behavior & Organization* 42:109-124.

[3] Dixit, A., and J. Londregan (1996), "The determinants of success of special interests in redistributive politics," *Journal of Politics* 58:1132-1155.

[4] Esteban, J.-M., and D. Ray (1994), "On the measurement of polarization," *Econometrica* 62:819-851.

[5] Fudenberg, D., and J. Tirole (2000), "Customer poaching and brand switching," *RAND Journal of Economics* 31:634-657.

[6] Grossman, G. M., and E. Helpman (1996), "Electoral competition and special interest politics," *Review of Economic Studies* 63:265-286.

[7] Judd, K. L. (1985), "The law of large numbers with a continuum of iid random variables," *Journal of Economic Theory* 35:19-25.

[8] Konrad, K. (2002), "Investment in the absence of property rights," *European Economic Review* 46:1521-1537.

[9] Laslier, J. F. (2002), "How two-party competition treats minorities," *Review of Economic Design* 7:297-307.

[10] Laslier, J. F., and N. Picard (2002), "Distributive politics and electoral competition," *Journal of Economic Theory* 103:106-130.

[11] Lee, B. (1997), "Markets with consumer switching benefits," Management Research Lab, Korea Telecom, mimeo.

[12] Lein, D. D. (1990), "Corruption and allocation efficiency," *Journal of Development Economics* 33:153-164.

[13] Lizzeri, A. (1999), "Budget deficits and redistributive politics," *Review of Economic Studies* 66:909-928.

[14] Lizzeri, A., and N. Persico (2001), "The provision of public goods under alternative electoral incentives," *American Economic Review* 91:225-239.

[15] Lizzeri, A., and N. Persico (2002), "The drawbacks of electoral competition," University of Pennsylvania, mimeo.

[16] Myerson, R. B. (1993), "Incentives to cultivate favored minorities under alternative electoral systems," *American Political Science Review* 87:856-869.

[17] Nelson, R. B. (1999), *An Introduction to Copulas*, (Springer, New York, NY).

[18] PBS (2004), "Persuaders," an Episode of Frontline, available at www.pbs.org/wgbh/pages/frontline/shows/persuaders/.

[19] Rodriguez, J. G., and R. Salas (2003), "Extended bi-polarization and inequality measures," in Bishop, J. A., and Y. Amiel (eds.) *Research on Economic Inequality*, Vol. 9 (Elsevier Science, New York, NY).

[20] Sahuguet, N., and N. Persico (2004), "Campaign spending regulation in a model of redistributive politics," *Economic Theory* forthcoming.

[21] Tocqueville, A. de (2004), *Democracy in America*, translated by Goldhammer, A. (Literary Classics of the United States, Inc., New York, NY).

[22] Wang, Y.-Q., and K.-Y. Tsui (2000), "Polarization orderings and new classes of polarization indices," *Journal of Public Economic Theory* 2:349-363.

[23] Wolfson, M. C. (1994), "When inequalities diverge," *American Economic Review* 84:353-358.

3.5 Appendix

Sahuguet and Persico (2004) establish an equivalence between the two-party model of redistributive politics and an appropriately chosen two-bidder all-pay auction. We now extend this result to establish an equivalence between the two-party model of redistributive politics with segmented loyal voters and an appropriately chosen set of two-bidder independent simultaneous all-pay auctions.

We begin by reviewing the characterization of n simultaneous two-bidder all-pay auctions with complete information. Let F_i^j represent bidder i's distribution of bids for auction j, and v_i^j represent the value of auction j for bidder i. Each bidder i's problem is

$$\max_{\{F_i^j\}_{j=1}^n} \sum_{j=1}^n \int_0^\infty \left[v_i^j F_{-i}^j(x) - x\right] dF_i^j$$

Since each auction is independent, the unique equilibrium is for each bidder to choose F_i^j as if auction j was the only auction. The case of a single all-pay auction with complete information is studied by Baye, Kovenock, de Vries (1996). Thus, for each auction j and bidder i we have the following three cases

$$\begin{array}{lll} If\ v_i^j > v_{-i}^j & F_i^j(x) = \frac{x}{v_{-i}^j} & x \in \left[0, v_{-i}^j\right] \\ If\ v_i^j = v_{-i}^j & F_i^j(x) = \frac{x}{v_i^j} & x \in \left[0, v_i^j\right] \\ If\ v_i^j < v_{-i}^j & F_i^j(x) = \left(\frac{v_{-i}^j - v_i^j}{v_{-i}^j}\right) + \frac{x}{v_{-i}^j} & x \in \left[0, v_i^j\right] \end{array}$$

In addition, without a binding cap on bids, there is no reason to construct an n-variate distribution function from these marginal distributions.[12]

Now consider two-party redistributive competition with segmented loyal voters, and assume that the parties face the budget constraint

$$\sum_{j \in A \cup S \cup B} m_j \int_0^\infty x dF_i^j \leq 1,$$

[12] Without a binding cap on bids, it is trivial to construct an n-variate distribution since any n-variate copula is sufficient. Given the Fréchet-Hoeffding bounds for n-variate copulas, the range of sufficient n-variate copulas is quite large. For this reason the n-variate joint distribution adds nothing to the problem analyzed here. See Nelson (1999) for an introduction to copulas.

where F_i^j represents party i's distribution of offers for voters in segment j and $m_j > 0$ is the measure of voters in segment j such that $\sum_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}} m_j = 1$. In the discussion that follows the notation for the intensity of loyal voter attachment is modified in the following way: for each segment $j \in \mathcal{A} \cup S \cup \mathcal{B}$ if $j = S$, the swing segment, or $j \in \mathcal{B}$, one of party B's loyal voter segments, then $\alpha_A^j = 0$, thus $\delta_A^j = 1$, and the same holds for α_B^k if $k \in \mathcal{A} \cup S$. Each party i's problem is

$$\max_{\{F_i^j\}_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}}} \sum_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}} m_j \int_0^\infty F_{-i}^j \left(\frac{x \delta_{-i}^j}{\delta_i^j} \right) dF_i^j$$

subject to the budget constraint $\sum_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}} m_j \int_0^\infty x dF_i^j(x) \leq 1$. The associated Lagrangian is

$$\max_{\{F_i^j\}_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}}} \sum_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}} \left[m_j \lambda_i \int_0^\infty \left[\frac{1}{\lambda_i} F_{-i}^j \left(\frac{x \delta_{-i}^j}{\delta_i^j} \right) - x \right] dF_i^j(x) \right] + \lambda_i$$

We can now proceed to the proof of the equivalence between the two-party model of redistributive politics with segmented loyal voters and an appropriately chosen set of two-bidder independent simultaneous all-pay auctions. In the discussion that follows, $\bar{s}_i^j$ and $\underline{s}_i^j$ are the upper and lower bounds of candidate i's distribution of offers in segment j.

Theorem 2: For each feasible distribution of voters' attachments to the political parties, there exists a one-to-one correspondence between the equilibria of the two-party model of redistributive politics with segmented loyal voters and the equilibria of a unique set of appropriately chosen two-bidder independent simultaneous all-pay auctions.

Proof: The proof, which is contained in the following lemmas, is instructive in that it establishes the uniqueness of the equilibrium given in Theorem 1.

The first three lemmas follow from lines drawn by Baye, Kovenock, and de Vries (1996).

Lemma 1: For each $j \in \mathcal{A} \cup S \cup \mathcal{B}$, $\frac{\bar{s}_{-i}^j \delta_i^j}{\delta_{-i}^j} = \bar{s}_i^j$.

Lemma 2: In any equilibrium $\{F_i^j, F_{-i}^j\}_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}}$, no F_i^j can place an atom in the half open interval $(0, \bar{s}_i^j]$.

Lemma 3: For each $j \in \mathcal{A}\cup\mathcal{S}\cup\mathcal{B}$ and for each $i \in \{A, B\}$, $\frac{1}{\lambda_i}F_{-i}^j\left(\frac{x\delta_{-i}^j}{\delta_i^j}\right) - x$ is constant $\forall\ x \in (0, \bar{s}_i^j]$.

The following lemma characterizes the relationship between λ_i and λ_{-i}.

Lemma 4: In equilibrium $\lambda_i = \lambda_{-i}$.

Proof: By way of contradiction suppose $\lambda_i \neq \lambda_{-i}$. In any equilibrium each party must use their entire budget, thus

$$\sum_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}} m_j \int_0^{\bar{s}_i^j} x dF_i^j(x) = \sum_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}} m_j \int_0^{\bar{s}_{-i}^j} x dF_{-i}^j(x) \tag{3.2}$$

But, from lemmas 2 and 3, it follows that

$$dF_i^j(x) = \lambda_{-i}\frac{\delta_{-i}^j}{\delta_i^j}dx \tag{3.3}$$

for all $x \in (0, \bar{s}_i^j]$, and

$$dF_{-i}^j(x) = \lambda_i\frac{\delta_i^j}{\delta_{-i}^j}dx \tag{3.4}$$

for all $x \in (0, \bar{s}_{-i}^j]$. Substituting equations 3 and 4 into equation 2, and applying lemma 1 we have

$$\lambda_{-i}\sum_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}} m_j \int_0^{\frac{\bar{s}_{-i}^j\delta_i^j}{\delta_{-i}^j}} x\frac{\delta_{-i}^j}{\delta_i^j}dx = \lambda_i \sum_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}} m_j \int_0^{\bar{s}_{-i}^j} x\frac{\delta_i^j}{\delta_{-i}^j}dx$$

which is a contradiction since

$$\sum_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}} m_j \int_0^{\frac{\bar{s}_{-i}^j\delta_i^j}{\delta_{-i}^j}} x\frac{\delta_{-i}^j}{\delta_i^j}dx = \sum_{j\in\mathcal{A}\cup\mathcal{S}\cup\mathcal{B}} m_j \int_0^{\bar{s}_{-i}^j} x\frac{\delta_i^j}{\delta_{-i}^j}dx$$

but $\lambda_i \neq \lambda_{-i}$. Q.E.D.

Let $\lambda \equiv \lambda_i = \lambda_{-i}$. The following lemma establishes the value of $\bar{s}_i^j$.

Lemma 5: $\bar{s}_i^j = \frac{\delta_i^j}{\lambda}$ $\forall$ i and j.

Proof: From lemmas 3 and 4, we know that for each party i and any segment j

$$\frac{1}{\lambda} F_{-i}^j \left(\frac{x \delta_{-i}^j}{\delta_i^j} \right) - x$$

is constant $\forall$ $x \in \left(0, \bar{s}_i^j\right]$. It then follows that party i would never use a strategy that provides offers in $\left(\frac{1}{\lambda}, \infty\right)$ since an offer of zero strictly dominates such a strategy. The result follows directly. Q.E.D.

The following lemma establishes that there exists a unique λ that satisfies the budget constraint.

Lemma 6: There exists a unique value for λ, and this value is $\frac{1 - \sum_{j \in \mathcal{A}} m_j \left(1 - \delta_A^j\right) - \sum_{k \in \mathcal{B}} m_k \left(1 - \delta_B^k\right)}{2} = \frac{1 - \sigma_A - \sigma_B}{2}$.

Proof: The budget constraint determines the unique value of λ. Thus, λ solves

$$\lambda \sum_{j \in \mathcal{A} \cup \mathcal{S} \cup \mathcal{B}} m_j \int_0^{\frac{\delta_i^j}{\lambda}} x \frac{\delta_{-i}^j}{\delta_i^j} dx = 1$$

Solving for λ we have that

$$\lambda = \frac{1 + \sum_{j \in \mathcal{A}} m_j \left(\delta_A^j - 1\right) + \sum_{k \in \mathcal{B}} m_k \left(\delta_B^k - 1\right)}{2} = \frac{1 - \sigma_A - \sigma_B}{2}$$

Q.E.D.

This completes the proof of Theorem 2.

To construct the unique Nash equilibrium of the redistributive politics game, note that the intensity of attachment parameters, $\alpha_i^j = 1 - \delta_i^j$, are isomorphic to differences in bidders' valuations in an all-pay auction. Thus, in each segment of voters loyal to party $-i$, party i places mass equal to $\frac{\frac{1}{\lambda} - \frac{1}{\lambda}\delta_{-i}}{\frac{1}{\lambda}} = 1 - \delta_{-i}$ at 0. Then letting $z \equiv \frac{1}{\lambda}$, the unique equilibrium is for party A to offer redistribution according to

$$\begin{array}{llr}
\forall\, j \in \mathcal{A} & F_A^j(x) = \frac{x}{z\delta_A^j} & x \in \left[0, z\delta_A^j\right] \\
 & F_A^S(x) = \frac{x}{z} & x \in [0, z] \\
\forall\, k \in \mathcal{B} & F_A^k(x) = \left(1 - \delta_B^k\right) + \frac{\delta_B^k x}{z} & x \in [0, z]
\end{array}$$

and for party B to offer redistribution according to

$$\begin{array}{llr}
\forall\, k \in \mathcal{B} & F_B^k(x) = \frac{x}{z\delta_B^k} & x \in \left[0, z\delta_B^k\right] \\
 & F_B^S(x) = \frac{x}{z} & x \in [0, z] \\
\forall\, j \in \mathcal{A} & F_B^j(x) = \left(1 - \delta_A^j\right) + \frac{\delta_A^j x}{z} & x \in [0, z]
\end{array}$$

where $z = \frac{1}{\lambda} = \frac{2}{1-\sigma_A-\sigma_B}$.

VITA

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